Graph signal processing - 101

- Undirected $G$ with adjacency matrix $W$
- Define a signal $x$ on the graph $\Rightarrow x_i = \text{Signal value at node } i$
- Associated with $G$ is the combinatorial graph Laplacian $L = D - W$
- $D$ is diagonal matrix where its elements are vertices degree
- Graph Laplacian is positive semidefinite so $L = \mathbf{V} \Lambda \mathbf{V}^T$
- $\mathbf{GSP} \rightarrow \text{Exploit structure encoded in } W \text{ or } L \text{ to process } x$
- $\text{Use GSP to learn the underlying } G \text{ or a meaningful network model}$
- Graph Fourier Transform (GFT): $x_i = \mathbf{V}^T x$ for undirected graphs
- $\text{Decompose } x \text{ into different modes of variation}$
- $\text{Inverse (i) GFT } x = \mathbf{V} \mathbf{x}$, eigenvectors as frequency atoms
- Total variation (or Dirichlet energy) of signal $x$ with respect to $L$
  $$TV(x) = x^T L x = \sum_{i,j} W_{ij} \left(x_i - x_j\right)^2 = \sum_{k=1}^N \lambda_k x_k^2$$
- For Laplacian eigenvectors $V = [v_1, \ldots, v_N] \Rightarrow TV(v_j) = \lambda_j$
- $0 = \lambda_1 < \lambda_2 \ldots \lambda_N$ can be viewed as frequencies

Discriminative graph learning

- Discriminative graph learning per class $c$
  $$\min_{W \in \mathbb{R}_+^{N \times N}} \left\{ \sum_{i=1}^{N} W_{ij} - \alpha \log(W_{ij}) + \beta (W_{ij} - \gamma)^2 \right\}$$
- $W_{ij} \in \{0, 1\} \Rightarrow \text{Capture the underlying graph topology (class } c \text{ structure)}$
- $\text{Discriminability to boost classification performance}$
- $\text{Logarithmic barrier forces positive degrees}$
- $\text{Penalize large edge-weights to control sparsity}$

Classification of network data

- labeled graph signals $X_c = \{x_j^{(c)}\}_{j=1}^p$, from $C$ different classes.
- Signals in each class possess a very distinctive structure
- Assumption: Class $c$ signals are smooth w.r.t. unknown $G_c(V, E_c)$
- Multiple linear subspace model
- Signals spanned by few Laplacian modes (GFT components)
- $\text{Like subspace clustering, but with supervision}$

Problem statement

Given training signals $X = \bigcup_{c=1}^C X_c$, learn discriminative graphs $W_c$, under smoothness priors to classify test signals via GFT projections.

Figures:

Figure 1: Significantly different connections between low and high for valence (left) and arousal (right).

Figure 2: Mean of eigenvectors magnitudes corresponding to low frequencies.

References