Sampling and Estimation in Network Graphs

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Network sampling

Network sampling and challenges

Background on statistical sampling theory

Network graph sampling designs

Estimation of network totals and group size

Estimation of degree distributions
Sampling network graphs

- Measurements often gathered **only from a portion** of a complex system
  - **Ex:** social study of high-school class vs. large corporation, Internet
  - Network graph → **sample** from a larger underlying network
- **Goal:** use sampled network data to infer properties of the whole system
  - Approach using principles of **statistical sampling theory**
- **Sampling in network contexts introduces various potential challenges**

### System under study

\[ G(V, E) \]

**Population graph**

\[ G^*(V^*, E^*) \]

**Available measurements**

\[ G^*(V^*, E^*) \]

**Sampled graph**

\[ G^*(V^*, E^*) \]

- \( G^* \) often a subgraph of \( G \) (i.e., \( V^* \subseteq V, E^* \subseteq E \)), but may not be
The fundamental problem

- Suppose a given graph characteristic or summary $\eta(G)$ is of interest
  - **Ex:** order $N_v$, size $N_e$, degree $d_v$, clustering coefficient $\text{cl}(G)$, . . .

- Typically impossible to recover $\eta(G)$ exactly from $G^*$
  - $\Rightarrow$ **Q:** Can we still form a useful estimate $\hat{\eta} = \hat{\eta}(G^*)$ of $\eta(G)$?

- **Plug-in estimator** $\hat{\eta} := \eta(G^*)$
  - Boils down to computing the characteristic of interest in $G^*$
  - Many familiar estimators in statistical practice are of this type
    - **Ex:** sample means, standard deviations, covariances, quantiles . . .

- Oftentimes $\eta(G^*)$ is a poor representation of $\eta(G)$
Example: Estimating average degreee

Let $G(V, E)$ be a network of protein interactions in yeast

$\Rightarrow$ Characteristic of interest is average degree

$$\eta(G) = \frac{1}{N_v} \sum_{i \in V} d_i$$

Here $N_v = 5,151$, $N_e = 31,201 \Rightarrow \eta(G) = 12.115$

Consider two sampling designs to obtain $G^*$

- First sample $n$ vertices $V^* = \{i_1, \ldots, i_n\}$ without replacement
- **Design 1**: For each $i \in V^*$, observe incident edges $(i, j) \in E$
- **Design 2**: Observe edge $(i, j)$ only when both $i, j \in V^*$

Estimate $\eta(G)$ by averaging the observed degree sequence $\{d_i^*\}_{i \in V^*}$

$$\eta(G^*) = \frac{1}{n} \sum_{i \in V^*} d_i^*$$
Random sample of \( n = 1,500 \) vertices, Designs 1 and 2 for edges

\[ \Rightarrow \] Process repeated for 10,000 trials \[ \Rightarrow \] histogram of \( \eta(G^*) \)

Under-estimate \( \eta(G) \) for Design 2, but Design 1 on target. Why?

- **Design 1**: sample vertex degree explicitly, i.e., \( d_i^* = d_i \)
- **Design 2**: (implicitly) sample vertex degree with bias, i.e., \( d_i^* \approx \frac{n}{N_v} d_i \)
Improving estimation accuracy

- In order to do better we need to incorporate the effects of
  - Random sampling; and/or
  - Measurement error

- Sampling design, topology of $G$, nature of $\eta(\cdot)$ all critical

- Model-based inference $\rightarrow$ Likelihood-based and Bayesian paradigms

- Design-based methods $\rightarrow$ Statistical sampling theory
  - Assume observations made without measurement error
  - Only source of randomness $\rightarrow$ sampling procedure

- **Ex:** Estimating average degree
  - Under Design 2 the estimate is biased, with mean of only 3.528
  - Adjusting $\eta(G^*)$ upward by a factor $\frac{N_v}{n} = 3.434$ yields 12,115

- Will see how statistical sampling theory justifies this correction
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Statistical sampling theory

- Suppose we have a population $\mathcal{U} = \{1, \ldots, N_u\}$ of $N_u$ units
  - Ex: People, animals, objects, vertices, . . .

- A value $y_i$ is associated with each unit $i \in \mathcal{U}$
  - Ex: Height, age, gender, infected, membership, . . .

- Typical interest in the population totals $\tau$ and averages $\mu$

$$
\tau := \sum_{i \in \mathcal{U}} y_i \quad \text{and} \quad \mu := \frac{1}{N_u} \sum_{i \in \mathcal{U}} y_i = \frac{1}{N_u} \tau
$$

- Basic sampling theory paradigm oriented around these steps:
  - **S1:** Randomly sample $n$ units $S = \{i_1, \ldots, i_n\}$ from $\mathcal{U}$
  - **S2:** Observe the value $y_{ik}$ for $k = 1, \ldots, n$
  - **S3:** Form an unbiased estimator $\hat{\mu}$ of $\mu$, i.e., $\mathbb{E}[\hat{\mu}] = \mu$
  - **S4:** Evaluate or estimate the variance $\text{var} [\hat{\mu}]$
Def: For given sampling design, the inclusion probability $\pi_i$ of unit $i$ is

$$\pi_i := P(\text{unit } i \text{ belongs in the sample } S)$$

Simple random sampling (SRS): $n$ units sampled uniformly from $U$

Without replacement: $i_1$ chosen from $U$, $i_2$ from $U \setminus \{i_1\}$, and so on

$\Rightarrow$ There are $\binom{N_u}{n}$ such possible samples of size $n$

$\Rightarrow$ There are $\binom{N_u-1}{n-1}$ samples which include a given unit $i$

The inclusion probability is

$$\pi_i = \frac{\binom{N_u-1}{n-1}}{\binom{N_u}{n}} = \frac{n}{N_u}$$
Sample mean estimator

- Definition of sample mean estimator

\[ \hat{\mu} = \frac{1}{n} \sum_{i \in S} y_i \]

- Using indicator RVs \( \mathbb{I} \{ i \in S \} \) for \( i \in \mathcal{U} \), where \( \mathbb{E} \left[ \mathbb{I} \{ i \in S \} \right] = \pi_i \)

\[ \Rightarrow \mathbb{E} \left[ \hat{\mu} \right] = \mathbb{E} \left[ \frac{1}{n} \sum_{i \in S} y_i \right] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{N_u} y_i \mathbb{I} \{ i \in S \} \right] = \frac{1}{n} \sum_{i=1}^{N_u} y_i \mathbb{E} \left[ \mathbb{I} \{ i \in S \} \right] = \frac{1}{n} \sum_{i=1}^{N_u} y_i \pi_i \]

- SRS without replacement \( \rightarrow \) unbiased because \( \pi_i = \frac{n}{N_u} \)

- Unequal probability sampling
  - More common than SRS, especially with networks. (More soon)
  - Sample mean can be a poor (i.e., biased) estimator for \( \mu \)
Horvitz-Thompson estimation for totals

- **Idea**: weighted average using inclusion probabilities as weights

### Horvitz-Thompson (HT) estimator

\[ \hat{\mu}_\pi = \frac{1}{N_u} \sum_{i \in S} \frac{y_i}{\pi_i} \quad \text{and} \quad \hat{\tau}_\pi = N_u \hat{\mu}_\pi \]

- Remedies the bias problem

\[ \mathbb{E} [\hat{\mu}_\pi] = \frac{1}{N_u} \sum_{i=1}^{N_u} \frac{y_i}{\pi_i} \mathbb{E} [\mathbb{I}\{i \in S\}] = \frac{1}{N_u} \sum_{i=1}^{N_u} y_i = \mu \]

⇒ Size of the population \( N_u \) assumed known

⇒ Broad applicability, but \( \pi_i \) may be difficult to compute
Horvitz-Thompson estimator variance

- **Def:** Joint inclusion probability $\pi_{ij}$ of units $i$ and $j$ is

  $$\pi_{ij} := P(\text{units } i \text{ and } j \text{ belong in the sample } S)$$

- If inclusion of units $i$ and $j$ are independent events $\Rightarrow \pi_{ij} = \pi_i \pi_j$

- **Ex:** Simple random sampling without replacement yields

  $$\pi_{ij} = \frac{n(n-1)}{N_u(N_u-1)}$$

- Variance of the HT estimator:

  $$\text{var} [\hat{\tau}_\pi] = \sum_{i\in U} \sum_{j\in U} y_i y_j \left( \frac{\pi_{ij}}{\pi_i \pi_j} - 1 \right), \quad \text{var} [\hat{\mu}_\pi] = \frac{\text{var} [\hat{\tau}_\pi]}{N_u^2}$$

  $\Rightarrow$ Typically estimated in an unbiased fashion from the sample $S$
Probability proportional to size sampling

- Unequal probability sampling
  \[ n \text{ units selected w.r.t. a distribution } \{p_1, \ldots, p_{N_u}\} \text{ on } \mathcal{U} \]
  \[ \Rightarrow \text{Uniform sampling: special case with } p_i = \frac{1}{N_u} \text{ for all } i \in \mathcal{U} \]

- Probability proportional to size (PPS) sampling
  \[ \Rightarrow \text{Probabilities } p_i \text{ proportional to a characteristic } c_i \]
  \[ \text{Ex: households chosen by drawing names from a database} \]

- If sampling with replacement, PPS inclusion probabilities are
  \[ \pi_i = 1 - (1 - p_i)^n, \text{ where } p_i = \frac{c_i}{\sum_k c_k} \]

- Joint inclusion probabilities for variance calculations
  \[ \pi_{ij} = \pi_i + \pi_j - [1 - (1 - p_i - p_j)^n] \]
Estimation of group size

- So far implicitly assumed $N_u$ known $\rightarrow$ Often not the case!
  
  Example: endangered animal species, people at risk of rare disease

- Special population total often of interest is the group size

  $$N_u = \sum_{i \in U} 1$$

- Suggests the following HT estimator of $N_u$

  $$\hat{N}_u = \sum_{i \in S} \pi_i^{-1}$$

  $\Rightarrow$ Infeasible, since knowledge of $N_u$ needed to compute $\pi_i$
Capture-recapture estimator

- Capture-recapture estimators overcome HT limitations in this setting

- Two rounds of SRS without replacement ⇒ Two samples $S_1, S_2$

**Round 1:** Mark all units in sample $S_1$ of size $n_1$ from $U$
  - Ex: tagging a fish, noting the ID number...
  - All units in $S_1$ are returned to the population

**Round 2:** Obtain a sample $S_2$ of size $n_2$ from $U$

**Capture-recapture estimator of $N_u$**

$$\hat{N}_u := \frac{n_2}{m} n_1, \text{ where } m := |S_1 \cap S_2|$$

- Factor $m/n_2$ indicative of marked fraction of the overall population
  ⇒ Can derive using model-based arguments as an ML estimator
Common network graph sampling designs

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Graph sampling designs

Q: What are common designs for sampling a network graph $G$?
A: Will see a few examples, along with their inclusion probabilities $\pi_i$

Graph-based sampling designs

⇒ Two inter-related classes of units, vertices $i$ and edges $(i,j)$

Often two stages

- Selection among one class of units (e.g., vertices)
- Observation of units from the other class (e.g., edges)

Inclusion probabilities offer insight into the nature of the designs

⇒ Central to HT estimators of network graph characteristics $\eta(G)$
Induced subgraph sampling

\( S \) Sample \( n \) vertices \( V^* = \{ i_1, \ldots, i_n \} \) without replacement (SRS)

\( O \) Observe edges \((i, j) \in E^*\) only when both \( i, j \in V^* \) (induced by \( V^* \))

- **Ex:** construction of contact networks in social network research
- Vertex and edge inclusion probabilities are uniformly equal to:

\[
\pi_i = \frac{n}{N_v} \quad \text{and} \quad \pi_{\{i,j\}} = \frac{n(n-1)}{N_v(N_v-1)}
\]
Consider a complementary design to induced subgraph sampling

S) Sample $n$ edges $E^*$ without replacement (SRS)

O) Observe vertices $i \in V^*$ incident to those selected edges in $E^*$

▶ Ex: construction of sampled telephone call graphs
Inclusion probabilities

- For incident subgraph sampling, edge inclusion probabilities are
  \[ \pi_{\{i,j\}} = \frac{n}{N_e} \]

- Vertex in \( V^* \) if any one or more of its incident edges are sampled
  \[ \pi_i = P(\text{vertex } i \text{ is sampled}) \]
  \[ = 1 - P(\text{no edge incident to } i \text{ is sampled}) \]
  \[ = \begin{cases} 
  1 - \left( \frac{N_e - d_i}{\binom{N_e}{n}} \right), & \text{if } n \leq N_e - d_i \\
  1, & \text{if } n > N_e - d_i
  \end{cases} \]

- Vertices included with unequal probs. that depend on their degrees
  \( \Rightarrow \) Probability proportional to size (degree) sampling of vertices
  \( \Rightarrow \) Requires knowledge of \( N_e \) and degree sequence \( \{d_i\}_{i \in V^*} \)
Snowball sampling

S) Sample $n$ vertices $V_0^* = \{i_1, \ldots, i_n\}$ without replacement (SRS)

O1) Observe edges $E_0^*$ incident to each $i \in V_0^*$, forming the initial wave

O2) Observe neighbors $\mathcal{N}(V_0^*)$ of $i \in V_0^*$, i.e., $V_1^* = \mathcal{N}(V_0^*) \cap (V_0^*)^c$

▶ Iterate to a desired number of e.g., $k$ waves, or until $V_k^*$ empty

$\Rightarrow G^*$ has $V^* = V_0^* \cup V_1^* \cup \ldots \cup V_k^*$, and their incident edges

▶ Ex: ‘spiders’ or ‘crawlers’ to discover the WWW’s structure
Difficult to compute inclusion probabilities beyond a single wave
⇒ Single-wave snowball sampling reduces to star sampling

Unlabeled: $V^* = V_0^*$ and $E^* = E_0^*$ their incident edges
- **Ex:** Count all co-authors of $n$ sampled authors
- Vertex inclusion probabilities are simply $\pi_i = n/N_v$

Labeled: $V^* = V_0^* \cup (\mathcal{N}(V_0^*) \cap (V_0^*)^c)$ and $E^* = E_0^*$
- **Ex:** Count and identify all co-authors of $n$ sampled authors
- Vertex inclusion probabilities can be shown to look like

$$
\pi_i = \sum_{L \subseteq \mathcal{N}_i} (-1)^{|L|+1} P(L), \text{ where } P(L) = \binom{N_v - |L|}{n-|L|} \binom{N_v}{n}
$$

- Denoted by $\mathcal{N}_i$ the neighborhood of vertex $i$ (including $i$ itself)
Link tracing

- **Link-tracing designs**
  - Select an initial sample of vertices $V^*_S$
  - Trace edges (links) from $V^*_S$ to another set of vertices $V^*_T$

- **Snowball sampling**: special case where all incident edges are traced

- May be infeasible to follow all incident edges to a given vertex
  - **Ex:** lack of recollection/deception in social contact networks

- **Path sampling designs**
  - Source nodes $V^*_S = \{s_1, \ldots, s_{n_S}\} \subset V$
  - Target nodes $V^*_T = \{t_1, \ldots, t_{n_T}\} \subset V \setminus V^*_S$
  - Traverse and measure the path between each pair $(s_i, t_j)$
  - **Ex:** Traceroute Internet studies, Milgram’s “Six Degrees” experiment
Traceroute sampling

- Trace shortest paths from each source to all targets

- Vertex and edge inclusion probabilities roughly [Dall’Asta et al ’06]:
  \[
  \pi_i \approx 1 - (1 - \rho_S - \rho_T)e^{-\rho_S\rho_Tc_{Be}(i)} \quad \text{and} \quad \pi\{i,j\} \approx 1 - e^{-\rho_S\rho_Tc_{Be}(\{i,j\})}
  \]

- Source and target sampling fractions \( \rho_S := n_S/N_\nu \) and \( \rho_T := n_T/N_\nu \)
  \( \Rightarrow \) Induces PPS sampling, size given by betweenness centralities
Estimation of totals in network graphs

- Network sampling and challenges
- Background on statistical sampling theory
- Network graph sampling designs
- Estimation of network totals and group size
- Estimation of degree distributions
Network summaries as totals

- Various graph summaries $\eta(G)$ are expressible in terms of totals $\tau$

  **Average degree:** Let $\mathcal{U} = V$ and $y_i = d_i$, then $\eta(G) = \bar{d} \propto \sum_{i \in V} d_i$

  **Graph size:** Let $\mathcal{U} = E$ and $y_{ij} = 1$, then $\eta(G) = N_e = \sum_{(i,j) \in E} 1$

  **Betweenness centrality:** Let $\mathcal{U} = V^{(2)}$ (unordered vertex pairs) and $y_{ij} = \mathbb{I} \{ k \in \mathcal{P}_{(i,j)} \}$. For unique shortest $i - j$ paths $\mathcal{P}_{(i,j)}$, then

  $$\eta(G) = c_{Be}(k) = \sum_{(i,j) \in V^{(2)}} \mathbb{I} \{ k \in \mathcal{P}_{(i,j)} \}$$

  **Clustering coefficient:** Let $\mathcal{U} = V^{(3)}$ (unordered vertex triples), then

  $$\eta(G) = cl(G) = 3 \times \frac{\text{total number of triangles}}{\text{total number of connected triples}}$$

- Often such totals can be obtained from sampled $G^*$ via HT estimation
Vertex totals

- Vertex totals are of the form $\tau = \sum_{i \in V} y_i$, averages are $\tau / N_v$
  - **Ex:** average degree where $y_i = d_i$
  - **Ex:** nodes with characteristic $C$, where $y_i = \mathbb{I} \{ i \in C \}$

- Given a sample $V^* \subseteq V$, the HT estimator for vertex totals is
  \[
  \hat{\tau}_\pi = \sum_{i \in V^*} \frac{y_i}{\pi_i}
  \]

  $\Rightarrow$ Variance expressions carry over, let $U = V$ and $V^*$ for estimates

- **Inclusion probabilities** $\pi_i$ depend on network sampling design
  $\Rightarrow$ Sampling also influences whether $y_i$ is observable, e.g., $y_i = d_i$
Totals on vertex pairs

- Quantity $y_{ij}$ corresponding to vertex pairs $(i, j) \in V^{(2)}$ of interest
  $\Rightarrow$ Totals $\tau = \sum_{(i,j)\in V^{(2)}} y_{ij}$ become relevant

- Ex: graph size $N_e$ and betweenness $c_{Be}(k)$ where $y_{ij} = \mathbb{I}\{k \in \mathcal{P}_{(i,j)}\}$

- Ex: shared gender in friendship network, average dissimilarity

- The HT estimator in this context is

$$\hat{\tau}_\pi = \sum_{(i,j)\in V^{(2)}*} \frac{y_{ij}}{\pi_{ij}}$$

$\Rightarrow$ Edge totals a special case, when $y_{ij} \neq 0$ only for $(i, j) \in E$

- Variance expression increasingly complicated, namely

$$\text{var} [\hat{\tau}_\pi] = \sum_{(i,j)\in V^{(2)}} \sum_{(k,l)\in V^{(2)}} y_{ik}y_{kl} \left( \frac{\pi_{ijkl}}{\pi_{ij}\pi_{kl}} - 1 \right)$$

$\Rightarrow$ Depends on inclusion probabilities $\pi_{ijkl}$ of vertex quadruples
Consider estimating $N_e$ as an edge total, i.e.,

$$N_e = \sum_{(i,j) \in E} 1 = \sum_{(i,j) \in V(2)} A_{ij}$$

**Bernoulli sampling (BS):** $\mathbb{I}\{i \in V^*\} \sim \text{Ber}(p)$ i.i.d. for all $i \in V$

$\Rightarrow$ Edges $E^*$ obtained via induced subgraph sampling $\Rightarrow \pi_{ij} = p^2$

**The HT estimator of $N_e$ is**

$$\hat{N}_e = \sum_{(i,j) \in V(2)^*} \frac{A_{ij}}{\pi_{ij}} = p^{-2} N_e^*$$

$\Rightarrow$ Scales up the empirically observed edge total $N_e^*$ by $p^{-2} > 1$

**Variance can be shown to take the form [Frank '77]**

$$\text{var} \left[ \hat{N}_e \right] = (p^{-1} - 1) \sum_{i \in V} d_i^2 + (p^{-2} - 2p^{-1} + 1) N_e$$
Example: Estimating network size (cont.)

- **Protein network:** $N_v = 5,151$, $N_e = 31,201$
  - BS of vertices with $p = 0.1$ and $p = 0.3$
  - Process repeated for 10,000 trials ⇒ histogram of $\hat{N}_e$

- Average of $\hat{N}_e$ was 31,116 and 31,203 ⇒ **Unbiasedness supported**
  - Mean and variability of $\hat{se}$ shrinks with $p$ (larger sample)
Example: Estimating clustering coefficient

- Average clustering coefficient $\text{cl}(G)$ can be expressed as

$$
\text{cl}(G) = 3 \times \frac{\tau_\triangle(G)}{\tau_3(G)}
$$

- Involves the quotient of two totals on vertex triples

$$
\tau = \sum_{(i,j,k) \in V(3)} y_{ijk} \Rightarrow \hat{\tau}_\pi = \sum_{(i,j,k) \in V(3)^*} \frac{y_{ijk}}{\pi_{ijk}}
$$

- Total number of triangles $\tau_\triangle(G)$, where

$$
y_{ijk} = A_{ij}A_{jk}A_{ki}
$$

- Total number of connected triples $\tau_3(G)$, where

$$
y_{ijk} = A_{ij}A_{jk}(1 - A_{ki}) + A_{ij}(1 - A_{jk})A_{ki} + (1 - A_{ij})A_{jk}A_{ki}
$$
Example: Estimating clustering coefficient (cont.)

- **Protein network:** $\tau_\Delta(G) = 44,858$, $\tau_3(G) \approx 1M$, and $cl(G) = 0.1179$
  - BS of vertices with $p = 0.2$
  - Induced subgraph sampling of edges
  - Process repeated for 10,000 trials \(\Rightarrow\) histogram of $\hat{cl}(G)$

![Histograms of estimates](image)

- Unbiased HT estimators $\hat{\tau}_\Delta = p^{-3}\tau_\Delta(G^*)$ and $\hat{\tau}_3 = p^{-3}\tau_3(G^*)$
  - Plug-in estimator $\hat{cl}(G) = 3\hat{\tau}_\Delta/\hat{\tau}_3$ results in $\hat{cl}(G) = cl(G^*)$\(\Rightarrow\) Quite accurate with mean 0.1191 and $\hat{se}$ of 0.0251
Horvitz-Thompson framework fairly straightforward in its essence

Success in network sampling and estimation rests on interaction among
  a) Sampling design;
  b) Measurements taken; and
  c) Total to be estimated

Three basic elements must be present in the problem
  1) Network summary statistic $\eta(G)$ expressible as total;
  2) Values $y$ either observed, or obtainable from measurements; and
  3) Inclusion probabilities $\pi$ computable for the sampling design

Unfortunately, often not all three are present at the same time . . .
Recall our first example on estimation of average degree $\frac{1}{N_v} \sum_{i \in V} d_i$

- **Design 1**: Unlabeled star sampling, observes degrees $d_i$, $i \in V^*$
- **Design 2**: Induced subgraph sampling, does not observe degrees

Average degree is a scaling of a vertex total ($N_v$ known)

$\Rightarrow$ HT estimation applicable so long as $y_i = d_i$ observed

True for unlabeled star sampling, and since $\pi_i = n/N_v$ we have

$$\hat{\mu}_{St} = \frac{\hat{r}_{St}}{N_v}, \text{ where } \hat{r}_{St} = \sum_{i \in V_*^{St}} \frac{d_i}{n/N_v}$$

We do not observe $d_i$ under induced subgraph sampling

$\Rightarrow$ Not amenable to HT estimation as vertex total for this design
Example: Estimating average degree (cont.)

- Identity $\mu = \frac{2N_e}{N_v} \Rightarrow$ Tackle instead as estimation of network size $N_e$

- For induced subgraph sampling $\pi_{ij} = \frac{n(n-1)}{N_v(N_v-1)}$, so HT estimator is

$$\hat{N}_{e,IS} = \sum_{(i,j) \in V(2)\ast} \frac{A_{ij}}{n(n-1)/[N_v(N_v-1)]} = \frac{N_v(N_v-1)}{n(n-1)} N_{e,IS}^*$$

$\Rightarrow$ Desired unbiased estimator for the average degree is

$$\hat{\mu}_{IS} = \frac{2\hat{N}_{e,IS}}{N_v}$$

- Estimators under both designs can be compared by writing them as

$$\hat{\mu}_{St} = \frac{2N_{e,St}^*}{n} \quad \text{and} \quad \hat{\mu}_{IS} = \frac{2N_{e,IS}^*}{n} \cdot \frac{N_v - 1}{n - 1}$$

$\Rightarrow$ Design 1: uses the identity $\mu = \frac{2N_e}{N_v}$ on $G_{St}^*$

$\Rightarrow$ Design 2: same but inflated by $\frac{N_v-1}{n-1}$, compensates $d_{i,IS}^* < d_i$
Assuming that $N_v$ is known may not be on safe grounds

⇒ Human or animal groups too mobile or elusive to count accurately
⇒ All Web pages or Internet routers are too massive and dispersed

Often estimating $N_v$ may well be the prime objective

If vertex SRS or BS feasible, could sample twice ‘marking’ in between
⇒ Facilitates usage of capture-recapture estimators ‘off-the-shelf’

If sampling infeasible, or capture-recapture performs poorly
⇒ Develop estimators of $N_v$ tailored to the graph sampling at hand
Hidden population: individuals do not wish to expose themselves
- Ex: humans of socially sensitive status, such as homeless
- Ex: involved in socially sensitive activities, e.g., drugs, prostitution

Such groups are often small ⇒ Estimating their size is challenging

Snowball sampling used to estimate the size of hidden populations

Sampling a hidden population

- Directed graph $G(V, E)$, $V$ the members of the hidden population
  - Graph describing willingness to identify other members
  - Arc $(i, j)$ when ask individual $i$, mentions $j$ as a member

- Graph $G^*$ obtained via one-wave snowball sampling, i.e., $V^* = V^*_0 \cup V^*_1$
  - Initial sample $V^*_0$ obtained via BS from $V$ with probability $p_0$

- Consider the following random variables (RVs) of interest
  - $N = |V^*_0|$: size of the initial sample
  - $M_1$: number of arcs among individuals in $V^*_0$
  - $M_2$: number of arcs from individuals in $V^*_0$ to individuals in $V^*_1$

- Snowball sampling yields measurements $n, m_1, \text{ and } m_2$ of these RVs
Method of moments estimator

- **Method of moments**: equate moments to sample counterparts

\[
\mathbb{E}[N] = \mathbb{E} \left[ \sum_i \mathbb{I} \{i \in V_0^*\} \right] = N_v p_0 = n
\]

\[
\mathbb{E}[M_1] = \mathbb{E} \left[ \sum_j \sum_{i \neq j} \mathbb{I} \{i \in V_0^*\} \mathbb{I} \{j \in V_0^*\} A_{ij} \right] = N_e p_0^2 = m_1
\]

\[
\mathbb{E}[M_2] = \mathbb{E} \left[ \sum_j \sum_{i \neq j} \mathbb{I} \{i \in V_0^*\} \mathbb{I} \{j \not\in V_0^*\} A_{ij} \right] = N_e p_0 (1 - p_0) = m_2
\]

- Expectation w.r.t. randomness in selecting the sample \(V_0^*\). Solution:

\[
\hat{N}_v = n \left( \frac{m_1 + m_2}{m_1} \right)
\]

⇒ Size of initial sample inflated by estimate of the sampling rate
Estimation of degree distributions

Network sampling and challenges

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Estimation of degree distributions
Estimation of other network characteristics

- Classical sampling theory rests heavily on Horvitz-Thompson framework
  - Scope limited to network totals
  - Q: Other network summaries, e.g., degree distributions?

- Findings on the effect of sampling on observed degree distributions:
  - Highly unrepresentative of actual degree distributions; and
  - Unhelpful to characterizing heterogeneous distributions

- Ex: Internet traceroute sampling [Lakhina et al’ 03]
  - Broad degree distribution in $G^*$, while concentrated in $G$

- Ex: Sampling protein-protein interaction networks [Han et al’ 05]
  - Power-law exponent estimate from $G^*$ underestimates $\alpha$ in $G$
Impact of sampling on degree distribution

- Let $N(d)$ denote the number of vertices with degree $d$ in $G$
  - Let $N^*(d)$ be the counterpart in a sampled graph $G^*$
  - Introduce vectors $\mathbf{n} = [N(0), \ldots, N(d_{\text{max}})]^\top$ and likewise $\mathbf{n}^*$

- Under a variety of sampling designs, it holds that
  \[ \mathbb{E} [\mathbf{n}^*] = \mathbf{Pn} \]
  - Matrix $\mathbf{P}$ depends fully on the sampling, not $G$ itself
  - Expectation w.r.t. randomness in selecting the sample $G^*$

An inverse problem

- Recall the identity $\mathbb{E} [n^*] = Pn$ $\Rightarrow$ Face a linear inverse problem
- Unbiased estimator of the degree distribution $n$
  \[ \hat{n}_{\text{naive}} = P^{-1}n^* \]
- While natural, two problems with this simple solution
  $\Rightarrow$ Matrix $P$ typically not invertible in practice; and
  $\Rightarrow$ Non-negativity of the solution is not guaranteed
- We actually have an ill-posed linear inverse problem
Performance of naive estimator

- Erdős-Renyi graph with $N_v = 100$ and $N_e = 500$
  - BS of vertices with $p = 0.6$
  - Induced subgraph sampling of edges
Penalized least-squares formulation

- Constrained, penalized, weighted least-squares [Zhang et al '14]

\[
\min_n (P_n - n^*)^\top C^{-1} (P_n - n^*) + \lambda \text{pen}(n)
\]

s. to \(N(d) \geq 0, \ d = 0, 1, \ldots, d_{\text{max}},\)

\[
\sum_{d=1}^{d_{\text{max}}} N(d) = N_v
\]

⇒ Matrix \(C\) denotes the covariance of \(n^*\)

⇒ Functional \(\text{pen}(n)\) penalizes complexity in \(n\), tuned by \(\lambda\)

- Constraints

⇒ Non-negativity of degree counts

⇒ Total degree counts equal the number of vertices

⇒ Smoothness: \(\text{pen}(n) = \|Dn\|^2, \ D\) differentiating operator
Application to online social networks

- Communities from online social networks Orkut and LiveJournal
  - BS of vertices with $p = 0.3$
  - Induced subgraph sampling of edges

- **True**, sampled, and **estimated** degree distribution
Glossary

- Enumeration and sampling
- Population graph
- Sampled graph
- Plug-in estimator
- Sampling design
- Sample with(out) replacement
- Design-based methods
- Averages and totals
- Inclusion probability
- Simple random sampling
- Bernoulli sampling
- Unequal probability sampling
- Horvitz-Thompson estimator
- Probability proportional to size sampling
- Capture-recapture estimator
- Induced subgraph sampling
- Incident subgraph sampling
- Snowball and star sampling
- Traceroute sampling
- Hidden population
- Ill-posed inverse problem
- Penalized least squares