Degrees, Power Laws and Popularity

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Degree distributions

Power-law degree distributions

Visualizing and fitting power laws

Popularity and preferential attachment
Descriptive analysis of network characteristics

- Given a network graph representation of a complex system
  \[ \Rightarrow \text{Structural properties of } G \text{ key to system-level understanding} \]

Example

- **Q1:** Underpinning of various types of basic social dynamics?
  - **A:** Study vertex triplets (triads) and patterns of ties among them

- **Q2:** How can we formalize the notion of ‘importance’ in a network?
  - **A:** Define measures of individual vertex (or group) centrality

- **Q3:** Can we identify communities and cohesive subgroups?
  - **A:** Formulate as a graph partitioning (clustering) problem

- Characterization of individual vertices/edges and network cohesion
  - Social network analysis, math, computer science, statistical physics
Def: The degree $d_v$ of vertex $v$ is its number of incident edges.

Degree sequence arranges degrees in non-decreasing order.

In figure, vertices degrees shown in red, e.g., $d_1 = 2$ and $d_5 = 3$.

Graph’s degree sequence is 2,2,2,3,3,4.

In general, the degree sequence does not uniquely specify the graph.

High-degree vertices are likely to be influential, central, prominent.
Degree distribution

- Let $N(d)$ denote the number of vertices with degree $d$
  \[ \Rightarrow \text{Fraction of vertices with degree } d \text{ is } P(d) := \frac{N(d)}{N_v} \]

- **Def:** The collection \( \{P(d)\}_{d \geq 0} \) is the degree distribution of $G$
  - Histogram formed from the degree sequence (bins of size one)

- $P(d)$ = probability that randomly chosen node has degree $d$
  \[ \Rightarrow \text{Summarizes the local connectivity in the network graph} \]
Q: What about patterns of association among nodes of given degrees?
A: Define the two-dimensional analogue of a degree distribution

Prob. of random edge having incident vertices with degrees \((d_1, d_2)\)
A simple random graph model

- **Def:** The Erdős-Rényi random graph model $G_{n,p}$
  - Undirected graph with $n$ vertices, i.e., of order $N_v = n$
  - Edge $(u, v)$ present with probability $p$, independent of other edges

- **Simulation** is easy: draw $\binom{n}{2}$ i.i.d. Bernoulli($p$) RVs

**Example**

- Three realizations of $G_{10, \frac{1}{6}}$. The size $N_e$ is a random variable
Degree distribution of $G_{n,p}$

- **Q**: Degree distribution $P(d)$ of the Erdős-Renyi graph $G_{n,p}$?

- Define $\mathbb{I}\{(v, u)\} = 1$ if $(v, u) \in E$, and $\mathbb{I}\{(v, u)\} = 0$ otherwise.

  \[ \Rightarrow \text{Fix } v. \text{ For all } u \neq v, \text{ the indicator RVs are i.i.d. } \text{Bernoulli}(p) \]

- Let $D_v$ be the (random) degree of vertex $v$. Hence,

  \[ D_v = \sum_{u \neq v} \mathbb{I}\{(v, u)\} \]

  \[ \Rightarrow D_v \text{ is binomial with parameters } (n - 1, p) \text{ and} \]

  \[ P(d) = P(D_v = d) = \binom{n - 1}{d} p^d (1 - p)^{(n-1)-d} \]

- In words, the probability of having exactly $d$ edges incident to $v$

  \[ \Rightarrow \text{Same for all } v \in V, \text{ by independence of the } G_{n,p} \text{ model} \]
Behavior for large $n$

- **Q:** How does the degree distribution look like for a large network?

- Recall $D_v$ is a sum of $n - 1$ i.i.d. Bernoulli($p$) RVs
  
  $\Rightarrow$ Central Limit Theorem: $D_v \sim \mathcal{N}(np, np(1 - p))$ for large $n$

- Makes most sense to increase $n$ with fixed $\mathbb{E}[D_v] = (n - 1)p = \mu$
  
  $\Rightarrow$ Law of rare events: $D_v \sim \text{Poisson}(\mu)$ for large $n$
Law of rare events

- Substituting \( p = \mu / n \) in the binomial PMF yields

\[
P_n(d) = \frac{n!}{(n-d)!d!} \left( \frac{\mu}{n} \right)^d \left( 1 - \frac{\mu}{n} \right)^{n-d}
\]

\[
= \frac{n(n-1) \ldots (n-d+1)}{n^d} \frac{\mu^d}{d!} \left( 1 - \frac{\mu}{n} \right)^n \left( 1 - \frac{\mu}{n} \right)^d
\]

- In the limit, red term is \( \lim_{n \to \infty} (1 - \mu / n)^n = e^{-\mu} \)

- Black and blue terms converge to 1. Limit is the Poisson PMF

\[
\lim_{n \to \infty} P_n(d) = 1 \frac{\mu^d}{d!} \frac{e^{-\mu}}{1} = e^{-\mu} \frac{\mu^d}{d!}
\]

- Approximation usually called “law of rare events”
  - Individual edges happen with small probability \( p = \mu / n \)
  - The aggregate (degree, number of edges), though, need not be rare
The $G_{n,p}$ model and real-world networks

- For large graphs, $G_{n,p}$ suggests $P(d)$ with an exponential tail
  $\Rightarrow$ Unlikely to see degrees spanning several orders of magnitude

- Concentrated distribution around the mean $\mathbb{E}[D_v] = (n - 1)p$

- Q: Is this in agreement with real-world networks?
Degree distributions of the WWW analyzed in [Broder et al '00]

⇒ Web a digraph, study both in- and out-degree distributions

- Majority of vertices naturally have small degrees
  ⇒ Nontrivial amount with orders of magnitude higher degrees
The topology of the AS-level Internet studied in [Faloutsos \(^3\) ’99]

Right-skewed degree distributions also found for router-level Internet
More heavy-tailed degree distributions found in [Barabasi-Albert '99]

- These heterogeneous, diffuse degree distributions are not exponential
Power laws

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Log-log plots show roughly a linear decay, suggesting the power law

\[ P(d) \propto d^{-\alpha} \Rightarrow \log P(d) = C - \alpha \log d \]

- Power-law exponent (negative slope) is typically \( \alpha \in [2, 3] \)
- Normalization constant \( C \) is mostly uninteresting
- Power laws often best followed in the tail, i.e., for \( d \geq d_{\text{min}} \)
The scale-free property

The minimum degree, $a$, affects the size of its hubs. To answer this we calculate the expected maximum degree, $d_{\text{max}}$. The genetic network in a human cell has approximately 20,000 vertices, which represents the expected size of the largest hub in a network.

Erdős-Rényi's Poisson degree distribution exhibits a sharp cutoff at $k = 0.15$, while power laws upper bound exponential tails for large enough $d$.

$P(d) \propto d^{-2.1}$

Erdős-Renyi’s Poisson degree distribution exhibits a sharp cutoff

$\Rightarrow$ Power laws upper bound exponential tails for large enough $d$
Scale-free networks

- **Scale-free network**: degree distribution with power-law tail
  - Name motivated for the scale-invariance property of power laws

- **Def**: A scale-free function \( f(x) \) satisfies \( f(ax) = bf(x) \), for \( a, b \in \mathbb{R} \)

**Example**

- **Power-law functions** \( f(x) = x^{-\alpha} \) are scale-free since

\[
  f(ax) = (ax)^{-\alpha} = a^{-\alpha}f(x) = bf(x), \text{ where } b := a^{-\alpha}
\]

- **Exponential functions** \( f(x) = c^x \) are not scale-free because

\[
f(ax) = c^{ax} = (c^x)^a = f^a(x) \neq bf(x), \text{ except when } a = b = 1\]

- **No ‘characteristic scale’ for the degrees.** More soon

  \( \Rightarrow \) Functional form of the distribution is invariant to scale
Power-law distributions are ubiquitous

- Power-law distributions widespread beyond networks [Clauset et al ’07]
The power-law degree distribution $P(d) = C d^{-\alpha}$ is a PMF, hence

$$1 = \sum_{d=0}^{\infty} P(d) = \sum_{d=0}^{\infty} C d^{-\alpha} \Rightarrow C = \frac{1}{\sum_{d=0}^{\infty} d^{-\alpha}}$$

Often a power law is only valid for the tail $d \geq d_{\text{min}}$, hence

$$C = \frac{1}{\sum_{d=d_{\text{min}}}^{\infty} d^{-\alpha}} \approx \frac{1}{\int_{d_{\text{min}}}^{\infty} x^{-\alpha} dx} = (\alpha - 1) d_{\text{min}}^{\alpha-1}$$

⇒ Sound approximation since $P(d)$ varies slowly for large $d$

The normalized power-law degree distribution is

$$P(d) = \frac{\alpha - 1}{d_{\text{min}}} \left( \frac{d}{d_{\text{min}}} \right)^{-\alpha}, \quad d \geq d_{\text{min}}$$
Often convenient to treat degrees as real valued, i.e., \( d \in \mathbb{R}^+ \)

Define a power-law PDF for the tail of the degree distribution as

\[
p(d) = \frac{\alpha - 1}{d_{\text{min}}} \left( \frac{d}{d_{\text{min}}} \right)^{-\alpha}, \quad d \geq d_{\text{min}}
\]

⇒ A valid PDF, already showed that \( \int_{d_{\text{min}}}^{\infty} p(x) \, dx = 1 \)

⇒ Convergence of the integral requires \( \alpha > 1 \)

Ex: Probability that a random node has degree exceeding 100 is

\[
P(D_v > 100) = \int_{100}^{\infty} \frac{\alpha - 1}{d_{\text{min}}} \left( \frac{x}{d_{\text{min}}} \right)^{-\alpha} \, dx = \left( \frac{100}{d_{\text{min}}} \right)^{1-\alpha}
\]
Q: What is the $m$-th moment of a power-law distributed RV?

From the definition of moment and the power-law PDF one has

$$
\mathbb{E}[D_v^m] = \int_{d_{min}}^{\infty} x^m p(x) dx = \frac{\alpha - 1}{d_{min}^{1-\alpha}} \left[ \frac{x^{m+1-\alpha}}{m + 1 - \alpha} \right]_{d_{min}}^{\infty}
$$

\Rightarrow Convergence of the integral requires $m + 1 < \alpha$

For real-world networks, typically $\alpha \in (2, 3)$ so

$$
\mathbb{E}[D_v] = \left( \frac{\alpha - 1}{\alpha - 2} \right) d_{min} < \infty \quad \text{and} \quad \mathbb{E}[D_v^m] = \infty, \quad m \geq 2
$$

In particular, the second moment and variance are infinite

\Rightarrow Consistent with variability and heterogeneity of degrees
Revisiting the scale-free property

- A measure of scale of a RV is its standard deviation $\sigma$.

![Graph showing the distribution of degrees in a network]

**Large random network** $G_{n,p}$
- Randomly chosen node has degree $d = \mu \pm \sqrt{\mu}$. The scale is $\mu$.

**Scale-free network**
- Randomly chosen node has degree $d = \mu \pm \infty$. There is no scale.
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Visualizing power-law degree distributions

- A simple histogram may be problematic for visualizing $P(d)$
  - Use log-log scale to warp probabilities and widespread degrees

- Large statistical fluctuations (‘noise’) in the tail for large $d$
  - With bins of size one, high-degree counts are small
  - Makes sense to increase the bin size
Logarithmic binning

- Uniformly widening bins sacrifices resolution for small degrees
  ⇒ Use bins of different sizes in different parts of the histogram

- Logarithmic binning is widely used. The $n$-th bin is
  \[ a^{n-1} \leq d < a^n, \quad n = 1, 2, \ldots \]

  Ex: Common choice is $a = 2$, $n$-th bin has width $2^n - 2^{n-1} = 2^{n-1}$

- Normalize by the bin width. Wider bins will accrue higher counts
**Def**: The complementary cumulative distribution function (CCDF) is

\[ \bar{F}(d) = P(D_v \geq d) \]

⇒ Function \( \bar{F}(d) \) is the fraction of vertices with degree at least \( d \)

► For a power-law PDF, the CCDF also obeys a power law since

\[ P(D_v \geq d) = \int_d^{\infty} \frac{\alpha - 1}{d_{\text{min}}} \left( \frac{x}{d_{\text{min}}} \right)^{-\alpha} dx = \left( \frac{d}{d_{\text{min}}} \right)^{-(\alpha - 1)} \]

► If the PDF has exponent \( \alpha \), then CCDF \( \bar{F}(d) \) has exponent \( \alpha - 1 \)
Computing the CCDF

**Step 1:** List the degrees $d_v$ in descending order

**Step 2:** Assign ranks $r_v$ (from 1 to $N_v$) to vertices in that order

**Step 3:** The CCDF is the plot of $r_v/N_v$ versus degree $d_v$

- If degrees are repeated, CCDF is the largest value of $r_v/N_v$
- If $d$ not observed, $\bar{F}(d)$ = value for next (larger) observed degree
Plot the CCDF in a log-log scale and look for a straight-line behavior.

- Mitigates noise using cumulative frequencies (cf. raw frequencies)
- No binning needed ⇒ Avoids information loss as bins widen
Fitting power-law distributions

- Basic, yet nontrivial task is to estimate the exponent $\alpha$ from data

- A power law implies the linear model $\log P(d) = C - \alpha \log d + \epsilon$
  $\Rightarrow$ Natural to form the linear least-squares (LS) estimator
  $$\{ \hat{\alpha}, \hat{C} \} = \arg \min_{\alpha, C} \sum_i (\log P(d_i) - C + \alpha \log d_i)^2$$

- Simple, very popular, but not advisable for at least three reasons:
  1) Extremely noisy high-degree data, where the counts are the lowest
  2) Estimates are biased. The log transform distorts unevenly the errors
  3) If the power law is only valid in the tail, need to hand pick $d_{\min}$
A solution to the noise problem is to use the CCDF $\bar{F}(d)$

$\Rightarrow$ Cumulative frequencies smoothen the noise

Recall the CCDF follows a power law with exponent $\alpha - 1$

$\Rightarrow$ Can use a linear regression-based approach to find $\hat{\alpha}$, but . . .

Successive points in the CCDF plot are not mutually independent

$\Rightarrow$ (Ordinary) LS is not optimal for correlated errors
Maximum-likelihood estimator

- Suppose \( \{d_i\}_{i=1}^{N_v} \) are independent and follow a power law. MLE of \( \alpha \)?

  \[ f(d; \alpha) = \frac{\alpha^{-1}}{d_{\text{min}}} \left( \frac{d}{d_{\text{min}}} \right)^{-\alpha}, \quad d \geq d_{\text{min}} \]

- The log-likelihood function is (up to constants independent of \( \alpha \))

  \[ \ell_{N_v}(\alpha) = \sum_{i=1}^{N_v} \log f(d_i; \alpha) \propto N_v \log (\alpha - 1) - \alpha \sum_{i=1}^{N_v} \log \left( \frac{d_i}{d_{\text{min}}} \right) \]

- The MLE \( \hat{\alpha} \) (a.k.a. Hill estimator) solves the equation

  \[ \left. \frac{\partial \ell_{N_v}(\alpha)}{\partial \alpha} \right|_{\alpha=\hat{\alpha}} = \frac{N_v}{\hat{\alpha} - 1} - \sum_{i=1}^{N_v} \log \left( \frac{d_i}{d_{\text{min}}} \right) = 0 \]

- The solution is

  \[ \hat{\alpha} = 1 + \left[ \frac{1}{N_v} \sum_{i=1}^{N_v} \log \left( \frac{d_i}{d_{\text{min}}} \right) \right]^{-1} \]
Hill plot of ML estimates

Q: How can we go around hand-picking the value of $d_{\text{min}}$?

1) Rank-order degrees to obtain the sequence $d_{(1)} \leq \ldots \leq d_{(N_v)}$

2) For each $k \in \{1, \ldots, N_v - 1\}$ let $d_{\text{min}} = d_{(N_v-k)}$. The MLEs are

$$\hat{\alpha}(k) = 1 + \left[ \frac{1}{k} \sum_{i=0}^{k-1} \log \left( \frac{d_{(N_v-i)}}{d_{(N_v-k)}} \right) \right]^{-1}$$

3) Draw and examine the Hill plot of $\hat{\alpha}(k)$ versus $k$

- If a power law is credible, the Hill plot should ‘settle down’
  
  ⇒ Identify stable $\hat{\alpha}$ for a wide range of (intermediate) $k$ values

Q: Why focus on values on the intermediate range?

- Small $k$: Inaccurate estimation due to limited data
- Large $k$: Bias if power law is only valid in the tail
Example: Internet and protein interaction data

Power law is appropriate

Power law is inappropriate

Sharp decay in \( \hat{\alpha} \) suggests a simple power-law model is inappropriate.
Example: Flickr data

- Flickr social network: $N_v \approx 0.6M$, $N_e \approx 3.5M$ [Leskovec et al '08]

- Good fit to a power law with exponential cutoff $\bar{F}(d) \propto d^{-\alpha} e^{-\beta d}$
Popularity and preferential attachment

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Popularity as a network phenomenon

- **Popularity** is a phenomenon characterized by extreme imbalances
  - How can we quantify these imbalances? Why do they arise?

- Basic models of network behavior can be very insightful
  - Result of coupled decisions, correlated behavior in a population
Preferential attachment model

- Simple model for the creation of e.g., links among Web pages
- Vertices are created one at a time, denoted 1, \ldots, N_v
- When node \( j \) is created, it makes a single arc to \( i, 1 \leq i < j \)
- Creation of \((j, i)\) governed by a probabilistic rule:
  - With probability \( p \), \( j \) links to \( i \) chosen uniformly at random
  - With probability \( 1 - p \), \( j \) links to \( i \) with probability \( \propto d^\text{in}_i \)
- The resulting graph is directed, each vertex has \( d^\text{out}_v = 1 \)
- Preferential attachment model leads to “rich-gets-richer” dynamics
  - Arcs formed preferentially to (currently) most popular nodes
  - Prob. that \( i \) increases its popularity \( \propto i \)’s current popularity
Theorem

The preferential attachment model gives rise to a power-law in-degree distribution with exponent \( \alpha = 1 + \frac{1}{1-p} \), i.e.,

\[
P(d_{in} = d) \propto d^{-(1+\frac{1}{1-p})}
\]

- **Key:** “\( j \) links to \( i \) with probability \( \propto d_{in}^{i} \)” equivalent to copying, i.e., “\( j \) chooses \( k \) uniformly at random, and links to \( i \) if \( (k, i) \in E \)”

- **Reflect:** Copy other’s decision vs. independent decisions in \( G_{n,p} \)

- As \( p \to 0 \) \( \Rightarrow \) Copying more frequent \( \Rightarrow \) Smaller \( \alpha \to 2 \)
  - **Intuitive:** more likely to see extremely popular pages (heavier tail)
Continuous approximation

- In-degree \( d_{i}^{\text{in}}(t) \) of node \( i \) at time \( t \geq i \) is a RV. Two facts
  - **F1** Initial condition: \( d_{i}^{\text{in}}(i) = 0 \) since node \( i \) just created at time \( t = i \)
  - **F2** Dynamics of \( d_{i}^{\text{in}}(t) \): Probability that new node \( t + 1 > i \) links to \( i \) is
    \[
    P((t + 1, i) \in E) = p \times \frac{1}{t} + (1 - p) \times \frac{d_{i}^{\text{in}}(t)}{t}
    \]

- Will study a deterministic, continuous approximation to the model
  - Continuous time \( t \in [0, N] \)
  - Continuous degrees \( x_{i}^{\text{in}}(t) : [i, N] \mapsto \mathbb{R}^+ \) are deterministic

- Require in-degrees to satisfy the following growth equation
  \[
  \frac{dx_{i}^{\text{in}}(t)}{dt} = \frac{p}{t} + \frac{(1 - p)x_{i}^{\text{in}}(t)}{t}, \quad x_{i}^{\text{in}}(i) = 0
  \]
Solving the differential equation

1. Solve the first-order differential equation for $x_i^{in}(t)$ (let $q = 1 - p$)

$$\frac{dx_i^{in}}{dt} = \frac{p + qx_i^{in}}{t}$$

2. Divide both sides by $p + qx_i^{in}(t)$ and integrate over $t$

$$\int \frac{1}{p + qx_i^{in}} \frac{dx_i^{in}}{dt} dt = \int \frac{1}{t} dt$$

3. Solving the integrals, we obtain ($c$ is a constant)

$$\ln (p + qx_i^{in}) = q \ln (t) + c$$
Exponentiating and letting \( K = e^c \) we find

\[
\ln (p + qx_i^{in}(t)) = q \ln(t) + c \Rightarrow x_i^{in}(t) = \frac{1}{q} (Kt^q - p)
\]

To determine the unknown constant \( K \), use the initial condition

\[
0 = x_i^{in}(i) = \frac{1}{q} (Ki^q - p) \Rightarrow K = \frac{p}{iq}
\]

Hence, the deterministic approximation of \( d_i^{in}(t) \) evolves as

\[
x_i^{in}(t) = \frac{1}{q} \left( \frac{p}{iq} \times t^q - p \right) = \frac{p}{q} \left[ \left( \frac{t}{i} \right)^q - 1 \right]
\]
Q: At time $t$, what fraction $\bar{F}(d)$ of all nodes have in-degree $\geq d$?

Approximation: What fraction of all functions $x_i^{in}(t) \geq d$ by time $t$?

$$x_i^{in}(t) = \frac{p}{q} \left[ \left( \frac{t}{i} \right)^q - 1 \right] \geq d$$

Can be rewritten in terms of $i$ as

$$i \leq t \left[ \left( \frac{q}{p} \right) d + 1 \right]^{-1/q}$$

By time $t$ there are exactly $t$ nodes in the graph, so the fraction is

$$\bar{F}(d) = \left[ \left( \frac{q}{p} \right) d + 1 \right]^{-1/q} = 1 - F(d)$$
The degree distribution is given by the PDF $p(d)$.

Recall that the PDF, CDF and CCDF are related, namely

$$p(x) = \frac{dF(x)}{dx} = -\frac{d\bar{F}(x)}{dx}$$

Differentiating $\bar{F}(d) = \left(\frac{q}{p}\right)^d + 1^{-1/q}$ yields

$$p(d) = \frac{1}{p} \left[ \left(\frac{q}{p}\right)^d + 1 \right]^{-(1+\frac{1}{q})}$$

Showed $p(d) \propto d^{-(1+1/q)}$, a power law with exponent $\alpha = 1 + \frac{1}{1-p}$

⇒ Disclaimer: Relied on heuristic arguments

⇒ Rigorous, probabilistic analysis possible
Glossary

- Degree distribution
- Erdös-Renyi model
- Binomial distribution
- Law of rare events
- Right-skewed distribution
- Logarithmic scale
- Power law
- Exponential and heavy tails
- Scale-free network

- Characteristic scale
- Logarithmic binning
- Cumulative frequencies
- Hill estimator and plot
- Exponential cutoff
- Coupled decisions
- Preferential attachment model
- Rich-gets-richer phenomena
- Growth equation