Introduction to Random Processes

Arbitrages and pricing of stock options

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Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Arbitrage

- Bet on different events with each outcome paying a random return
- **Arbitrage**: possibility of devising a betting strategy that
  - Guarantees a positive return
  - No matter the combined outcome of the events
- Arbitrages often involve operating in two (or more) different markets
Ex: Booker 1  ⇒ Yankees win pays 1.5:1, Yankees loss pays 3:1

• Bet $x$ on Yankees and $y$ against Yankees. Guaranteed earnings?

  Yankees win: $0.5x - y > 0 \Rightarrow x > 2y$
  Yankees loose: $-x + 2y > 0 \Rightarrow x < 2y$

  ⇒ Arbitrage not possible. Notice that $\frac{1}{1.5} + \frac{1}{3} = 1$

Ex: Booker 2  ⇒ Yankees win pays 1.4:1, Yankees loss pays 3.1:1

• Bet $x$ on Yankees and $y$ against Yankees. Guaranteed earnings?

  Yankees win: $0.4x - y > 0 \Rightarrow x > 2.5y$
  Yankees loose: $-x + 2.1y > 0 \Rightarrow x < 2.1y$

  ⇒ Arbitrage not possible. Notice that $\frac{1}{1.4} + \frac{1}{3.1} > 1$
First condition on Booker 1 and second on Booker 2 are compatible

Bet $x$ on Yankees on Booker 1, $y$ against Yankees on Booker 2

Guaranteed earnings possible. Make e.g., $x = 2066$, $y = 1000$

Yankees win: $0.5 \times 2066 - 1000 = 33$
Yankees loose: $-2066 + 2.1 \times 1000 = 34$

⇒ Arbitrage possible. Notice that $1/(1.5) + 1/(3.1) < 1$

Sport bookies coordinate their odds to avoid arbitrage opportunities

⇒ Like card counting in casinos, arbitrage betting not illegal
⇒ But you will be banned if caught involved in such practices

If you plan on doing this, do it on, e.g., currency exchange markets
Let events on which bets are posted be \( k = 1, 2, \ldots, K \).

Let \( j = 1, 2, \ldots, J \) index possible joint outcomes.

- Joint realizations, also called “world realization”, or “world outcome”

If world outcome is \( j \), event \( k \) yields return \( r_{jk} \) per unit invested (bet).

Invest (bet) \( x_k \) in event \( k \) ⇒ return for world \( j \) is \( x_k r_{jk} \).

⇒ Bets \( x_k \) can be positive (\( x_k > 0 \)) or negative (\( x_k < 0 \)).

⇒ Positive = regular bet (buy). Negative = short bet (sell).

Total earnings ⇒ \( \sum_{k=1}^{K} x_k r_{jk} = x^T r_j \).

- Vectors of returns for outcome \( j \) ⇒ \( r_j := [r_{j1}, \ldots, r_{jK}]^T \) (given)
- Vector of bets ⇒ \( x := [x_1, \ldots, x_K]^T \) (controlled by gambler)
Notation in the sports betting example

**Ex:** Booker 1  ⇒  Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are $K = 2$ events to bet on
  - A Yankees’ win ($k = 1$) and a Yankees’ loss ($k = 2$)

- Naturally, there are $J = 2$ possible outcomes
  - Yankees won ($j = 1$) and Yankees lost ($j = 2$)

- **Q:** What are the returns?

  Yankees win ($j = 1$): $r_{11} = 0.5$,  $r_{12} = -1$
  Yankees loose ($j = 2$): $r_{21} = -1$,  $r_{22} = 2$

  ⇒ Return vectors are thus $r_1 = [0.5, -1]^T$ and $r_2 = [-1, 2]^T$

- Bet $x$ on Yankees and $y$ against Yankees, vector of bets $\mathbf{x} = [x, y]^T$
Arbitrage (clearly defined now)

- Arbitrage is possible if there exists investment strategy \( \mathbf{x} \) such that
  \[
  \mathbf{x}^T \mathbf{r}_j > 0, \quad \text{for all } j = 1, \ldots, J
  \]

- Equivalently, arbitrage is possible if
  \[
  \max_x \left( \min_j (\mathbf{x}^T \mathbf{r}_j) \right) > 0
  \]

- Earnings \( \mathbf{x}^T \mathbf{r}_j \) are the inner product of \( \mathbf{x} \) and \( \mathbf{r}_j \) (i.e., \( \perp \) projection)

\[ \Rightarrow \text{Positive earnings if angle between } \mathbf{x} \text{ and } \mathbf{r}_j < \pi/2 \left(90^\circ\right) \]
When is arbitrage possible?

- There is a line that leaves all $r_j$ vectors to one side
- There is not a line that leaves all $r_j$ vectors to one side

- Arbitrage possible
  - Prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
  
  $$E_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$
  
  does not exist

- Arbitrage not possible
  - There is prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
  
  $$E_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$

  - Think of $p_j$ as scaling factors
Arbitrage theorem

Have demonstrated the following result, called *arbitrage theorem*

⇒ Formal proof follows from duality theory in optimization

**Theorem**

*Given vectors of returns* \( \mathbf{r}_j \in \mathbb{R}^K \) *associated with random world outcomes* \( j = 1, \ldots, J \), *an arbitrage is not possible if and only if there exists a probability vector* \( \mathbf{p} = [p_1, \ldots, p_J]^T \) *with* \( p_j \geq 0 \) *and* \( \mathbf{p}^T \mathbf{1} = 1 \), *such that* \( \mathbb{E}_\mathbf{p}(\mathbf{r}) = 0 \). *Equivalently,*

\[
\max_x \left( \min_j (x^T \mathbf{r}_j) \right) \leq 0 \iff \sum_{j=1}^J p_j \mathbf{r}_j = 0
\]

- Prob. vector \( \mathbf{p} \) is **NOT** the prob. distribution of events \( j = 1, \ldots, J \)
Example: Arbitrages in sports betting

Ex: Booker 1 ⇒ Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are $K = 2$ events to bet on, $J = 2$ possible outcomes

- Q: What are the returns?

  Yankees win ($j = 1$): $r_{11} = 0.5$, $r_{12} = -1$
  Yankees loose ($j = 2$): $r_{21} = -1$, $r_{22} = 2$

  ⇒ Return vectors are thus $r_1 = [0.5, -1]^T$ and $r_2 = [-1, 2]^T$

- Arbitrage impossible if there is $0 \leq p \leq 1$ such that

  $$E_p(r) = p \times \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} + (1 - p) \times \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

  ⇒ Straightforward to check that $p = 2/3$ satisfies the equation
Consider a stock price $X(nh)$ that follows a geometric random walk:

$$X((n + 1)h) = X(nh)e^{\sigma\sqrt{h}Y_n}$$

- $Y_n$ is a binary random variable with probability distribution:

  $$P(Y_n = 1) = \frac{1}{2} \left(1 + \frac{\mu}{\sigma}\sqrt{h}\right), \quad P(Y_n = -1) = \frac{1}{2} \left(1 - \frac{\mu}{\sigma}\sqrt{h}\right)$$

  ⇒ As $h \to 0$, $X(nh)$ becomes geometric Brownian motion.

- Q: Are there arbitrage opportunities in trading this stock?
  ⇒ Too general, let us consider a narrower problem.
Consider the following investment strategy (stock flip):

**Buy:** Buy $1 in stock at time 0 for price \( X(0) \) per unit of stock

**Sell:** Sell stock at time \( h \) for price \( X(h) \) per unit of stock

Cost of transaction is $1. Units of stock purchased are \( 1/X(0) \)

⇒ Cash after selling stock is \( X(h)/X(0) \)

⇒ Return on investment is \( X(h)/X(0) - 1 \)

There are two possible outcomes for the price of the stock at time \( h \)

⇒ May have \( Y_0 = 1 \) or \( Y_0 = -1 \) respectively yielding

\[
X(h) = X(0)e^{\sigma \sqrt{h}}, \quad X(h) = X(0)e^{-\sigma \sqrt{h}}
\]

Possible returns are therefore

\[
r_1 = \frac{X(0)e^{\sigma \sqrt{h}}}{X(0)} - 1 = e^{\sigma \sqrt{h}} - 1, \quad r_2 = \frac{X(0)e^{-\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\sigma \sqrt{h}} - 1
\]
Present value of returns

- One dollar at time $h$ is not the same as 1 dollar at time 0
  ⇒ Must take into account the time value of money

- Interest rate of a risk-free investment is $\alpha$ continuously compounded
  ⇒ In practice, $\alpha$ is the money-market rate (time value of money)

- Prices have to be compared at their present value

- The present value (at time 0) of $X(h)$ is $X(h)e^{-\alpha h}$
  ⇒ Return on investment is $e^{-\alpha h}X(h)/X(0) - 1$

- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are
  \[
  r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{\sigma \sqrt{h}} - 1,
  \]
  \[
  r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma \sqrt{h}} - 1
  \]
No arbitrage condition

- Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1 - q)r_2 = 0$$

$\Rightarrow$ Arbitrage theorem in one dimension (only one bet, stock flip)

- Substituting $r_1$ and $r_2$ for their respective values

$$q \left( e^{-\alpha h} e^{\sigma \sqrt{h}} - 1 \right) + (1 - q) \left( e^{-\alpha h} e^{-\sigma \sqrt{h}} - 1 \right) = 0$$

- Can be easily solved for $q$. Expanding product and reordering terms

$$qe^{-\alpha h} e^{\sigma \sqrt{h}} + (1 - q)e^{-\alpha h} e^{-\sigma \sqrt{h}} = 1$$

- Multiplying by $e^{\alpha h}$ and grouping terms with a $q$ factor

$$q \left( e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}} \right) = e^{\alpha h} - e^{-\sigma \sqrt{h}}$$
No arbitrage condition (continued)

- Solving for $q$ finally yields
  \[ q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}} \]

- For small $h$ we have $e^{\alpha h} \approx 1 + \alpha h$ and $e^{\pm \sigma \sqrt{h}} \approx 1 \pm \sigma \sqrt{h} + \sigma^2 h/2$

- Thus, the value of $q$ as $h \to 0$ may be approximated as
  \[
  q \approx \frac{1 + \alpha h - \left(1 - \sigma \sqrt{h} + \sigma^2 h/2\right)}{1 + \sigma \sqrt{h} - \left(1 - \sigma \sqrt{h}\right)} = \frac{\sigma \sqrt{h} + (\alpha - \sigma^2/2) h}{2\sigma \sqrt{h}}
  \]
  \[
  = \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right)
  \]

- Approximation proves that at least for small $h$, then $0 < q < 1$
  \[ \Rightarrow \text{Arbitrage not possible} \]

- Also, suspiciously similar to probabilities of geometric random walk
  \[ \Rightarrow \text{Key observation as we’ll see next} \]
Risk neutral measure

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
No arbitrage condition on geometric random walk

- Stock prices $X(nh)$ follow geometric random walk (drift $\mu$, variance $\sigma^2$).
  
  $\Rightarrow$ Risk-free investment has return $\alpha$ (time value of money).

- Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

  $$q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}}$$

- Notice that $q$ satisfies the equation (which we’ll use later on)

  $$qe^{\sigma \sqrt{h}} + (1 - q)e^{-\sigma \sqrt{h}} = e^{\alpha h}$$

- **Q:** Can we have arbitrage using a more complex set of possible bets?
Consider the following general investment strategy:

**Observe:** Observe the stock price at times $h, 2h, \ldots, nh$

**Compare:** Is $X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n$?

**Buy:** If above answer is yes, buy stock at price $X(nh)$

**Sell:** Sell stock at time $mh$ ($m > n$) for price $X(mh)$

Possible bets are the observed values of the stock $x_1, x_2, \ldots, x_n$

$\Rightarrow$ There are $2^n$ possible bets

Possible outcomes are value at time $mh$ and observed values

$\Rightarrow$ There are $2^m$ possible outcomes
Explanation of general investment strategy

- There are $2^n$ possible bets:
  - Bet 1 = $n$ price increases in 1, \ldots, $n$
  - Bet 2 = price increases in 1, \ldots, $n - 1$ and price decrease in $n$
  - \ldots

- For each bet we have $2^{m-n}$ possible outcomes:
  - $m - n$ price increases in $n + 1, \ldots, m$
  - Price increases in $n + 1, \ldots, m - 1$ and price decrease in $m$
  - \ldots

<table>
<thead>
<tr>
<th></th>
<th>$X(h)$</th>
<th>$X(2h)$</th>
<th>$X(3h)$</th>
<th>$X(nh)$</th>
<th>$X((n+1)h)$</th>
<th>$X((n+2)h)$</th>
<th>$X(mh)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bet 1</td>
<td>$e^\sigma \sqrt{h}$</td>
<td>$e^{2\sigma \sqrt{h}}$</td>
<td>$e^{3\sigma \sqrt{h}}$</td>
<td>$e^{n\sigma \sqrt{h}}$</td>
<td>$X(nh)e^\sigma \sqrt{h}$</td>
<td>$X(nh)e^{2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{m\sigma \sqrt{h}}$</td>
</tr>
<tr>
<td>bet 2</td>
<td>$e^\sigma \sqrt{h}$</td>
<td>$e^{2\sigma \sqrt{h}}$</td>
<td>$e^{3\sigma \sqrt{h}}$</td>
<td>$e^{(n-2)\sigma \sqrt{h}}$</td>
<td>$X(nh)e^\sigma \sqrt{h}$</td>
<td>$X(nh)e^{2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{(m-2)\sigma \sqrt{h}}$</td>
</tr>
<tr>
<td>bet $2^n$</td>
<td>$e^{-\sigma \sqrt{h}}$</td>
<td>$e^{-2\sigma \sqrt{h}}$</td>
<td>$e^{-3\sigma \sqrt{h}}$</td>
<td>$e^{-n\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-m\sigma \sqrt{h}}$</td>
</tr>
</tbody>
</table>

- Table assumes $X(0) = 1$ for simplicity

outcomes per each bet
Define the prob. distribution $q$ over possible outcomes as follows

Start with a sequence of i.i.d. binary RVs $Y_n$, probabilities

$$P(Y_n = 1) = q, \quad P(Y_n = -1) = 1 - q$$

$\Rightarrow$ With $q = (e^{\alpha h} - e^{-\sigma \sqrt{h}})/(e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}})$ as in slide 18

Joint prob. distribution $q$ on $X(h), X(2h), \ldots, X(mh)$ from

$$X((n+1)h) = X(nh)e^{\sigma \sqrt{h}Y_n}$$

$\Rightarrow$ Recall this is NOT the prob. distribution of $X(nh)$

Will show that expected value of earnings with respect to $q$ is null

$\Rightarrow$ By arbitrage theorem, arbitrages are not possible
Return for given outcome

- Consider a time 0 unit investment in given arbitrary outcome

- Stock units purchased depend on the price $X(nh)$ at buying time

  \[
  \text{Units bought} = \frac{1}{X(nh)e^{-\alpha nh}}
  \]

  ⇒ Have corrected $X(nh)$ to express it in time 0 values

- Cash after selling stock given by price $X(mh)$ at sell time $m$

  \[
  \text{Cash after sell} = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}
  \]

- Return is then

  \[
  r(X(h), \ldots, X(mh)) = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1
  \]

  ⇒ Depends on $X(mh)$ and $X(nh)$ only
Expected return with respect to measure $q$

- Expected value of all possible returns with respect to $q$ is

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$

- Condition on observed values $X(h), \ldots, X(nh)$

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q(1:n) \left[ \mathbb{E}_q(n+1:m) \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \mid X(h), \ldots, X(nh) \right] \right]$$

- In innermost expectation $X(nh)$ is given. Furthermore, process $X$ is Markov, so conditioning on $X(h), \ldots, X((n-1)h)$ is irrelevant. Thus

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q(1:n) \left[ \frac{\mathbb{E}_q(n+1:m) \left[ X(mh) \mid X(nh) \right] e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$
Expected value of future values (measure $q$)

- Need to find expectation of future value $E_{q(n+1:m)} [X(mh) \mid X(nh)]$

- From recursive relation for $X(nh)$ in terms of $Y_n$ sequence

\[
X(mh) = X((m - 1)h) e^{\sigma \sqrt{h} Y_{m-1}} \\
= X((m - 2)h) e^{\sigma \sqrt{h} Y_{m-1}} e^{\sigma \sqrt{h} Y_{m-2}} \\
\vdots \\
= X(nh) e^{\sigma \sqrt{h} Y_{m-1}} e^{\sigma \sqrt{h} Y_{m-2}} \ldots e^{\sigma \sqrt{h} Y_n}
\]

- All the $Y_n$ are independent. Then, upon taking expectations

\[
E_{q(n+1:m)} [X(mh) \mid X(nh)] = X(nh) E \left[ e^{\sigma \sqrt{h} Y_{m-1}} \right] E \left[ e^{\sigma \sqrt{h} Y_{m-2}} \right] \ldots E \left[ e^{\sigma \sqrt{h} Y_n} \right]
\]

- Need to determine expectation of relative price change $E \left[ e^{\sigma \sqrt{h} Y_n} \right]$
The expected value of the relative price change $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$ is

$$\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} \Pr [Y_n = 1] + e^{-\sigma \sqrt{h}} \Pr [Y_n = -1]$$

According to definition of measure $q$, it holds

$$\Pr [Y_n = 1] = q, \quad \Pr [Y_n = -1] = 1 - q$$

Substituting in expression for $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$

$$\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} q + e^{-\sigma \sqrt{h}} (1 - q) = e^{\alpha h}$$

⇒ Follows from definition of probability $q$ [cf. slide 18]

Reweave the quilt:

(i) Use expected relative price change to compute expected future value
(ii) Use expected future value to obtain desired expected return
Reweave the quilt

- Plug $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\alpha h}$ into expression for expected future value

$$\mathbb{E}_{q(n+1:m)} \left[ X(mh) | X(nh) \right] = X(nh) e^{\alpha h} e^{\alpha h} \cdots e^{\alpha h} = X(nh) e^{\alpha (m-n) h}$$

- Substitute result into expression for expected return

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ \frac{X(nh) e^{\alpha (m-n) h} e^{-\alpha mh}}{X(nh) e^{-\alpha nh}} - 1 \right]$$

- Exponentials cancel out, finally yielding

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ 1 - 1 \right] = 0$$

$\Rightarrow$ Arbitrage not possible if $0 \leq q \leq 1$ exists (true for small $h$)
What if prices follow a geometric Brownian motion?

- Suppose stock prices follow a geometric Brownian motion, i.e.,
  \[ X(t) = X(0)e^{Y(t)} \]
  \[ ⇒ Y(t) \text{ Brownian motion with drift } \mu \text{ and variance } \sigma^2 \]

- Q: What is the no arbitrage condition?

- Approximate geometric Brownian motion by geometric random walk
  \[ ⇒ \text{Approximation arbitrarily accurate by letting } h \to 0 \]

- No arbitrage measure \( q \) exists for geometric random walk
  - This requires \( h \) sufficiently small
  - Notice that prob. distribution \( q = q(h) \) is a function of \( h \)

- Existence of the prob. distribution \( q := \lim_{h \to 0} q(h) \) proves that
  \[ ⇒ \text{Arbitrages are not possible in stock trading} \]
No arbitrage probability distribution

- Recall that as $h \to 0 \Rightarrow q \approx \frac{1}{2} \left( 1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h} \right)$

  $$\Rightarrow 1 - q \approx \frac{1}{2} \left( 1 - \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h} \right)$$

- Thus, measure $q := \lim_{h \to 0} q(h)$ is a geometric Brownian motion

  $$\Rightarrow \text{Variance } \sigma^2 \text{ (same as stock price)}$$

  $$\Rightarrow \text{Drift } \alpha - \sigma^2/2$$

- Measure showing arbitrage impossible a geometric Brownian motion

  $$\Rightarrow \text{Which is also the way stock prices evolve as } h \to 0$$

- Furthermore, the variance is the same as that of stock prices

  $$\Rightarrow \text{Different drifts } \Rightarrow \mu \text{ for stocks and } \alpha - \sigma^2/2 \text{ for no arbitrage}$$
Expected investment growth

- Compute expected return on an investment on stock $X(t)$
  - Buy 1 share of stock at time 0. Cash invested is $X(0)$
  - Sell stock at time $t$. Cash after sell is $X(t)$

- Expected value of cash after sell given $X(0)$ is

$$
\mathbb{E} \left[ X(t) \mid X(0) \right] = X(0)e^{(\mu + \sigma^2/2)t}
$$

- Alternatively, invest $X(0)$ risk free in the money market
  - Guaranteed cash at time $t$ is $X(0)e^{\alpha t}$

- Invest in stock only if $\mu + \sigma^2/2 > \alpha$  ⇒ “Risk premium” exists
Proof of expected return formula

- Stock prices follow a geometric Brownian motion $X(t) = X(0)e^{Y(t)}$
  $\Rightarrow Y(t)$ Brownian motion with drift $\mu$ and variance $\sigma^2$

- Q: What is the expected return $\mathbb{E}[X(t) | X(0)]$?

- Note first that $\mathbb{E}[X(t) | X(0)] = X(0)\mathbb{E}[e^{Y(t)} | X(0)]$

- Using that $Y(t)$ has independent increments

  $\mathbb{E}[e^{Y(t)} | X(0)] = \mathbb{E}[e^{Y(t)}]$

  $\Rightarrow$ Next we focus on computing $\mathbb{E}[e^{Y(t)}]$
Proof of expected return formula (cont.)

Since \( Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t) \)

\[
\mathbb{E} \left[ e^{Y(t)} \right] = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^y e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}} \, dy
\]

Completing the squares in the argument of the exponential we have

\[
y - \frac{(y - \mu t)^2}{2\sigma^2 t} = -y^2 + 2(\mu + \sigma^2)ty - \mu^2 t^2
\]

\[
= -\frac{(y - (\mu + \sigma^2)t)^2}{2\sigma^2 t} + \frac{2\mu\sigma^2 t^2 + \sigma^4 t^2}{2\sigma^2 t}
\]

The blue term does not depend on \( y \), red integral equals 1

\[
\mathbb{E} \left[ e^{Y(t)} \right] = e^{\left(\mu + \frac{\sigma^2}{2}\right)t} \times \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^{-\frac{(y-(\mu+\sigma^2)t)^2}{2\sigma^2 t}} \, dy = e^{\left(\mu + \frac{\sigma^2}{2}\right)t}
\]

Putting the pieces together, we obtain

\[
\mathbb{E} \left[ X(t) \mid X(0) \right] = X(0)\mathbb{E} \left[ e^{Y(t)} \right] = X(0)e^{(\mu+\sigma^2/2)t}
\]
Compute expected return as if \( q \) were the actual distribution

- Recall that \( q \) is **NOT** the actual distribution
- As before, cash invested is \( X(0) \) and cash after sale is \( X(t) \)

Expected cash value is different because prob. distribution is different

\[
\mathbb{E}_q [X(t) \mid X(0)] = X(0) e^{(\alpha-\sigma^2/2+\sigma^2/2)t} = X(0)e^{\alpha t}
\]

- Same return as risk-free investment regardless of parameters

Measure \( q \) is called **risk neutral measure**

- Risky stock investments yield same return as risk-free one
- “Alternate universe”, investors do not demand risk premiums

Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)

- Basis for Black-Scholes formula for option pricing
A continuous-time process $X(t)$ is a \textit{martingale} if for $t, s \geq 0$

$$\mathbb{E}[X(t + s) \mid X(u), 0 \leq u \leq t] = X(t)$$

⇒ Expected future value = present value, even given process history

Model of a fair, e.g., gambling game. \textit{Excludes winning strategies}

⇒ Even with prior info. of outcomes (cards drawn from the deck)

For risk-neutral measure $q$, time 0 prices $e^{-\alpha t}X(t)$ form a martingale

$$\mathbb{E}_q\left[e^{-\alpha(t+s)}X(t + s) \mid e^{-\alpha u}X(u), 0 \leq u \leq t\right] = e^{-\alpha t}X(t)$$

\textbf{Key principle:} stock price = expected discounted payoff

$$X(0) = \mathbb{E}_q\left[e^{-\alpha t}X(t) \mid X(0)\right]$$

⇒ Fair pricing, cannot devise a winning strategy (arbitrage)
Recall measure $q$ is a geometric Brownian motion $X(t) = e^{Y(t)}$

$\Rightarrow$ Variance $\sigma^2$ (same as stock price)

$\Rightarrow$ Drift $\alpha - \sigma^2/2$ 

Proof.

\[ \mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{Y(t+s)} \mid e^{-\alpha u} e^{Y(u)}, 0 \leq u \leq t \right] \]

\[ = \mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{Y(t+s)} \mid e^{-\alpha t} e^{Y(t)} \right] \]

\[ = \mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{[Y(t+s)-Y(t)]+Y(t)} \mid e^{-\alpha t} e^{Y(t)} \right] \]

$Y(t)$ is Markov

Add and subtract $Y(t)$

Independent increments

\[ = e^{-\alpha t} e^{Y(t)} \mathbb{E}_q \left[ e^{-\alpha s} e^{[Y(t+s)-Y(t)]} \right] \]

\[ = e^{-\alpha t} X(t) \mathbb{E}_q \left[ e^{-\alpha s} e^{Y(s)} \right] \]

Stationary increments

\[ = e^{-\alpha t} X(t) \]

\[ \mathbb{E}_q \left[ e^{Y(s)} \right] = e^{(\mu+\sigma^2/2)s} = e^{\alpha s} \]
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Options

- An **option** is a contract to buy shares of a stock at a future time
  - Strike time \( t = \) Convened time for stock purchase
  - Strike price \( K = \) Price at which stock is purchased at strike time

- At time \( t \), option holder may decide to
  - Buy a stock at strike price \( K = \) exercise the option
  - Do not exercise the option

- May buy option at time 0 for price \( c \)

- **Q:** How do we **determine the option’s worth**, i.e., price \( c \) at time 0?
- **A:** Given by the Black-Scholes formula for option pricing
Let $e^{\alpha t}$ be the compounding of a risk-free investment

Let $X(t)$ be the stock’s price at time $t$

$\Rightarrow$ Modeled as geometric Brownian motion, drift $\mu$, variance $\sigma^2$

Risk neutral measure $q$ is also a geometric Brownian motion

$\Rightarrow$ Drift $\alpha - \sigma^2/2$ and variance $\sigma^2$
Return of option investment

- At time $t$, the option’s worth depends on the stock’s price $X(t)$

- If stock’s price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  $\Rightarrow$ Option is worthless (better to buy stock at current price)

- Since had paid $c$ for the option at time 0, lost $c$ on this investment
  $\Rightarrow$ Return on investment is $r = -c$

- If stock’s price larger than strike price $\Rightarrow X(t) > K$
  $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$

- To obtain return express as time 0 values and subtract $c$
  $$r = e^{-\alpha t}(X(t) - K) - c$$

- May combine both in single equation $\Rightarrow r = e^{-\alpha t}(X(t) - K)_+ - c$
  $\Rightarrow (\cdot)_+ := \max(\cdot, 0)$ denotes projection onto positive reals $\mathbb{R}_+$
Select option price $c$ to prevent arbitrage opportunities

$$\mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)_+ - c \right] = 0$$

⇒ Expectation is with respect to risk neutral measure $q$

From above condition, the no-arbitrage price of the option is

$$c = e^{-\alpha t} \mathbb{E}_q \left[ (X(t) - K)_+ \right]$$

⇒ Source of Black-Scholes formula for option valuation

⇒ Rest of derivation is just evaluating $\mathbb{E}_q \left[ (X(t) - K)_+ \right]$

⇒ Same argument used to price any derivative of the stock’s price
Use fact that \( q \) is a geometric Brownian motion

- Let us evaluate \( E_q \left[ (X(t) - K)_+ \right] \) to compute option’s price \( c \)

- Recall \( q \) is a geometric Brownian motion \( \Rightarrow X(t) = X_0 e^{Y(t)} \)
  \( \Rightarrow X_0 = \) price at time 0
  \( \Rightarrow Y(t) \) BMD, \( \mu (= \alpha - \sigma^2/2) \) and variance \( \sigma^2 \)

- Can rewrite no arbitrage condition as

\[
c = e^{-\alpha t} E_q \left[ (X_0 e^{Y(t)} - K)_+ \right]
\]

- \( Y(t) \) is a Brownian motion with drift. Thus, \( Y(t) \sim N(\mu t, \sigma^2 t) \)

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi \sigma^2 t}} \int_{-\infty}^{\infty} (X_0 e^y - K)_+ e^{-(y-\mu t)^2/(2\sigma^2 t)} \, dy
\]
Evaluation of the integral

- Note that \((X_0e^{Y(t)} - K)_+ = 0\) for all values \(Y(t) \leq \log(K/X_0)\)

- Because integrand is null for \(Y(t) \leq \log(K/X_0)\) can write

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\log(K/X_0)}^{\infty} (X_0 e^y - K) e^{-(y-\mu t)^2/(2\sigma^2 t)} \, dy
\]

- Change of variables \(z = (y - \mu t)/\sqrt{\sigma^2 t}\). Associated replacements

  - Variable: \(y \Rightarrow \sqrt{\sigma^2 t}z + \mu t\)
  - Differential: \(dy \Rightarrow \sqrt{\sigma^2 t} \, dz\)
  - Integration limit: \(\log(K/X_0) \Rightarrow a := \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}}\)

- Option price then given by

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} (X_0 e^{\sqrt{\sigma^2 t}z+\mu t} - K) e^{-z^2/2} \, dz
\]
Separate in two integrals $c = e^{-\alpha t}(l_1 - l_2)$ where

\[ l_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{\sigma^2 t}z + \mu t} e^{-z^2/2} \, dz \]

\[ l_2 := \frac{K}{\sqrt{2\pi}} \int_a^\infty e^{-z^2/2} \, dz \]

Gaussian $\Phi$ function (ccdf of standard normal RV)

\[ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} \, dz \]

$\Rightarrow$ Comparing last two equations we have $l_2 = K\Phi(a)$

Integral $l_1$ requires some more work
Evaluation of the first integral

- Reorder terms in integral $I_1$

\[
I_1 := \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} X_0 e^{\sqrt{\sigma^2 t}z + \mu t} e^{-z^2/2} \, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{\sqrt{\sigma^2 tz} - z^2/2} \, dz
\]

- The exponent can be written as a square minus a “constant” (no $z$)

\[
-(z - \sqrt{\sigma^2 t})^2 / 2 + \sigma^2 t/2 = -z^2/2 + \sqrt{\sigma^2 t}z - \sigma^2 t/2 + \sigma^2 t/2
\]

- Substituting the latter into $I_1$ yields

\[
I_1 = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2 + \sigma^2 t/2} \, dz
\]

\[
= \frac{X_0 e^{\mu t + \sigma^2 t/2}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2} \, dz
\]
Evaluation of the first integral (continued)

- Change of variables $u = z - \sqrt{\sigma^2 t} \Rightarrow du = dz$ and integration limit

\[ a \Rightarrow b := a - \sqrt{\sigma^2 t} = \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}} - \sqrt{\sigma^2 t} \]

- Implementing change of variables in $l_1$

\[ l_1 = \frac{X_0 e^{\mu t + \sigma^2 t/2}}{\sqrt{2\pi}} \int_b^\infty e^{-u^2/2} \, du = X_0 e^{\mu t + \sigma^2 t/2} \Phi(b) \]

- Putting together results for $l_1$ and $l_2$

\[ c = e^{-\alpha t} (l_1 - l_2) = e^{-\alpha t} X_0 e^{\mu t + \sigma^2 t/2} \Phi(b) - e^{-\alpha t} K \Phi(a) \]

- For non-arbitrage stock prices (measure $\mathfrak{q}$) \( \Rightarrow \alpha = \mu + \sigma^2/2 \)

\( \Rightarrow \) Substitute to obtain Black-Scholes formula
**Black-Scholes**

- **Black-Scholes formula for option pricing.** Option cost at time 0 is

\[
c = X_0 \Phi(b) - e^{-\alpha t} K \Phi(a)
\]

\[\Rightarrow a := \log\left(\frac{K}{X_0}\right) - \mu t \quad \text{and} \quad b := a - \sqrt{\sigma^2 t}\]

- Note further that \( \mu = \alpha - \sigma^2/2 \). Can then write \( a \) as

\[
a = \frac{\log\left(\frac{K}{X_0}\right) - (\alpha - \sigma^2/2) t}{\sqrt{\sigma^2 t}}
\]

\[\Rightarrow X_0 = \text{stock price at time 0}, \quad \sigma^2 = \text{volatility of stock}\]

\[\Rightarrow K = \text{option’s strike price}, \quad t = \text{option’s strike time}\]

\[\Rightarrow \alpha = \text{benchmark risk-free rate of return (cost of money)}\]

- **Black-Scholes formula independent of stock’s mean tendency \( \mu \)**
Arbitrage
Investment strategy
Bets, events, outcomes
Returns and earnings
Arbitrage theorem
Geometric Brownian motion
Stock flip
Time value of money
Continuously-compounded interest
Present value

Risk-free investment
Expected return
Risk premium
Risk neutral measure
Pricing of derivatives
Stock option
Strike time and price
Option price
Stock volatility
Black-Scholes formula