Introduction to Random Processes

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Introductions

Class description and contents

Gambling
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Where? We meet in Wegmans Hall 1400 and online via Zoom
Meeting ID: 771 885 0098, passcode sent via email

When? Mondays and Wednesdays 4:50 pm to 6:05 pm

My office hours, Tuesdays at 11 am via Zoom (771 885 0098)
  Anytime, as long as you have something interesting to tell me

Class website
http://www.ece.rochester.edu/~gmateosb/ECE440.html
Three great TAs to help you with your homework

- **Narges Mohammadi**
  - Email: nmohamm4@ur.rochester.edu
  - Her office hours, Thursdays at 2 pm
  - Zoom: 381 188 3230

- **Shiyu Sun**
  - Email: ssun24@ur.rochester.edu
  - His office hours, Mondays at 10 am
  - Zoom: 470 562 9116
Teaching assistants

- Three great TAs to help you with your homework

- Saman Saboksayr
  - Email: ssaboksa@ur.rochester.edu
  - His office hours, Fridays at 10 am
  - Zoom: 236 855 9406
Prerequisites

(I) Probability theory
- Random (Stochastic) processes are collections of random variables
- Basic knowledge expected. Will review in the first five lectures

(II) Calculus and linear algebra
- Integrals, limits, infinite series, differential equations
- Vector/matrix notation, systems of linear equations, eigenvalues

(III) Programming in Matlab
- Needed for homework
  https://tech.rochester.edu/software/matlab/
- If you know programming you can learn Matlab in one afternoon
  ⇒ But it has to be one of this week’s afternoons
Homework, exams and grading

(I) Homework sets (10 in 15 weeks) worth 28 points
   ▶ Important and demanding part of this class
   ▶ Collaboration accepted, welcomed, and encouraged

(II) Midterm take-home examination on October 23 worth 36 points
   ▶ Usually an in-class, open notes exam. Change due to COVID-19

(III) Final take-home examination on December 13-15 worth 36 points
   ▶ Work independently. This time no collaboration, no discussion
   ▶ ECE 271 students get 10 free points
   ▶ At least 60 points are required for passing (C grade)
   ▶ B requires at least 75 points. A at least 92. No curve
      ⇒ Goal is for everyone to earn an A
Textbooks

- Good general reference for the class
  
  
  ⇒ Available online: http://www.library.rochester.edu/

- Also nice for topics including Markov chains, queuing models
  

- Both on reserve for the class in Carlson Library
Be nice

- I work hard for this course, expect you to do the same
  - If you come to class, be on time, pay attention, ask
  - Do all of your homework
  - Do not hand in as yours the solution of others (or mine)
  - Do not collaborate in the exams

- A little bit of (conditional) probability ...
- Probability of getting an E in this class is 0.04
- Probability of getting an E given you skip 4 homework sets is 0.7
  ⇒ I’ll give you three notices, afterwards, I’ll give up on you

- Come and learn. Useful down the road
Stop the spread

WEAR A MASK
Cover your nose and mouth with a face covering. Refrain from touching your face.

DR. CHAT BOT
Complete the Dr. Chat Bot screening online every day at uofr.us/chatbot.

WASH HANDS
Wash your hands often. Follow the CDC guidelines posted in restrooms.

STAY APART
Stay six feet apart. Adhere to physical distancing recommendations.

Learn more at rochester.edu/coronavirus-update
Class contents

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Gambling
Stochastic systems

- **Stochastic system**: Anything random that evolves in time
  \[ \text{Time can be discrete } n = 0, 1, 2 \ldots, \text{ or continuous } t \in [0, \infty) \]

- More formally, random processes assign a function to a random event
- Compare with “random variable assigns a value to a random event”

- Can interpret a random process as a collection of random variables
  \[ \text{Generalizes concept of random vector to functions} \]
  \[ \text{Or generalizes the concept of function to random settings} \]
A voice recognition system

- Random event $\sim$ word spoken. Random process $\sim$ the waveform
- Try the file speech_signals.m

```
"Hi"
```
```
"Good"
```
```
"Bye"
```
```
‘S’
```
Four thematic blocks

(I) Probability theory review (5 lectures)
- Probability spaces, random variables, independence, expectation
- Conditional probability: time $n + 1$ given time $n$, future given past ...
- Limits in probability, almost sure limits: behavior as $n \to \infty$ ...
- Common probability distributions (binomial, exponential, Poisson, Gaussian)

- Random processes are complicated entities
  ⇒ Restrict attention to particular classes that are somewhat tractable

(II) Markov chains (6 lectures)

(III) Continuous-time Markov chains (7 lectures)

(IV) Stationary random processes (8 lectures)

- Midterm covers up to Markov chains
Probability theory is a formalism to work with uncertainty
- Given a data-generating process, what are properties of outcomes?

Statistical inference deals with the inverse problem
- Given outcomes, what can we say on the data-generating process?
- CSC446 - Machine Learning, ECE442 - Network Science Analytics,
  CSC440 - Data Mining, ECE441 - Detection and Estimation Theory, . . .
Markov chains

- Countable set of states $1, 2, \ldots$. At discrete time $n$, state is $X_n$
- Memoryless (Markov) property
  \[ \Rightarrow \] Probability of next state $X_{n+1}$ depends on current state $X_n$
  \[ \Rightarrow \] But not on past states $X_{n-1}, X_{n-2}, \ldots$

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Tomorrow’s mood only affected by today’s mood
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

- Of interest: classification of states, ergodicity, limiting distributions
- Applications: Google’s PageRank, communication networks, queues, reinforcement learning, ...
Continuous-time Markov chains

- **Countable** set of states $1, 2, \ldots$ **Continuous-time** index $t$, state $X(t)$
  - Transition between states can happen at any time
  - **Markov**: Future independent of the past given the present

- Probability of changing state in an infinitesimal time $dt$

- **Of interest**: Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions

- **Applications**: Chemical reactions, queues, epidemic modeling, traffic engineering, weather forecasting, ...
Stationary random processes

- Continuous time \( t \), continuous state \( X(t) \), not necessarily Markov
- Prob. distribution of \( X(t) \) constant or becomes constant as \( t \) grows
  \( \Rightarrow \) System has a **steady state in a random sense**

- **Of interest:** Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density

- **Applications:** Black Scholes model for option pricing, radar, face recognition, noise in electric circuits, filtering and equalization, ...
Gambling

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Gambling
An interesting betting game

- There is a certain game in a certain casino in which ...
  - Your chances of winning are $p > 1/2$
- You place $1$ bets
  - (a) With probability $p$ you gain $1$; and
  - (b) With probability $1 - p$ you lose your $1$ bet
- The catch is that you either
  - (a) Play until you go broke (lose all your money)
  - (b) Keep playing forever
- You start with an initial wealth of $w_0$
- Q: Shall you play this game?
Let $t$ be a time index (number of bets placed)

Denote as $X(t)$ the outcome of the bet at time $t$

$\Rightarrow X(t) = 1$ if bet is won (w.p. $p$)

$\Rightarrow X(t) = 0$ if bet is lost (w.p. $1 - p$)

$X(t)$ is called a Bernoulli random variable with parameter $p$

Denote as $W(t)$ the player’s wealth at time $t$. Initialize $W(0) = w_0$

At times $t > 0$ wealth $W(t)$ depends on past wins and losses

$\Rightarrow$ When bet is won $W(t + 1) = W(t) + 1$

$\Rightarrow$ When bet is lost $W(t + 1) = W(t) - 1$

More compactly can write $W(t + 1) = W(t) + (2X(t) - 1)$

$\Rightarrow$ Only holds so long as $W(t) > 0$
Coding

\[ t = 0; \ w(t) = w_0; \ \max_t = 10^3; \ // \ \text{Initialize variables}\]
\[ \% \ \text{repeat while not broke up to time } \max_t \]
\[ \textbf{while} \ (w(t) > 0) \ \& \ (t < \max_t) \ \textbf{do} \]
\[ x(t) = \text{random}(\text{`bino'},1,p); \ \% \ \text{Draw Bernoulli random variable}\]
\[ \textbf{if} \ x(t) == 1 \ \textbf{then} \]
\[ w(t + 1) = w(t) + b; \ \% \ \text{If } x = 1 \ \text{wealth increases by } b\]
\[ \textbf{else} \]
\[ w(t + 1) = w(t) - b; \ \% \ \text{If } x = 0 \ \text{wealth decreases by } b\]
\[ \textbf{end} \]
\[ t = t + 1; \]
\[ \textbf{end} \]

\[ \triangleright \ \text{Initial wealth } w_0 = 20, \ \text{bet } b = 1, \ \text{win probability } p = 0.55 \]

\[ \triangleright \text{Q: Shall we play?} \]
She didn’t go broke. After $t = 1000$ bets, her wealth is $W(t) = 109$. Less likely to go broke now because wealth increased.
Two lucky players

- After \( t = 1000 \) bets, wealths are \( W_1(t) = 109 \) and \( W_2(t) = 139 \)

\[ \implies \text{Increasing wealth seems to be a pattern} \]
Ten lucky players

- Weights $W_j(t)$ after $t = 1000$ bets between 78 and 139

$\Rightarrow$ Increasing wealth is definitely a pattern
One unlucky player

- But this does not mean that all players will turn out as winners
  \[ \Rightarrow \text{The twelfth player } j = 12 \text{ goes broke} \]
One unlucky player

- But this does not mean that all players will turn out as winners.
  ⇒ The twelfth player $j = 12$ goes broke.
One hundred players

- All players (except for $j = 12$) end up with substantially more money
It is not difficult to find a line estimating the average of $W(t)$

\[ \bar{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t \] (recall $p = 0.55$)
Where does the average tendency come from?

- Assuming we do not go broke, we can write
  \[ W(t + 1) = W(t) + \left(2X(t) - 1\right), \quad t = 0, 1, 2, \ldots \]
  - The assumption is incorrect as we saw, but suffices for simplicity
- Taking expectations on both sides and using linearity of expectation
  \[ \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + \left(2\mathbb{E}[X(t)] - 1\right) \]
  - The expected value of Bernoulli \( X(t) \) is
    \[ \mathbb{E}[X(t)] = 1 \times P(X(t) = 1) + 0 \times P(X(t) = 0) = p \]
    - Which yields \( \Rightarrow \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + (2p - 1) \)
    - Applying recursively \( \Rightarrow \mathbb{E}[W(t + 1)] = w_0 + (2p - 1)(t + 1) \)
Recall the evolution of wealth \( W(t + 1) = W(t) + (2X(t) - 1) \)

View \( W(t + 1) \) as output of LTI system with random input \( 2X(t) - 1 \)

Recognize accumulator \( \Rightarrow W(t + 1) = w_0 + \sum_{\tau=0}^{t} \left( 2X(\tau) - 1 \right) \)

Useful, a lot we can say about sums of random variables

Filtering random processes in signal processing, communications, ...
Numerical analysis of simulation outcomes

- For a more accurate approximation analyze simulation outcomes
- Consider $J$ experiments. Each yields a wealth history $W_j(t)$
- Can estimate the average outcome via the sample average $\bar{W}_J(t)$

$$\bar{W}_J(t) := \frac{1}{J} \sum_{j=1}^{J} W_j(t)$$

- Do not confuse $\bar{W}_J(t)$ with $\mathbb{E}[W(t)]$
  - $\bar{W}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[W(t)]$ is a property of the random variable $W(t)$
  - We will see later that for large $J$, $\bar{W}_J(t) \to \mathbb{E}[W(t)]$
Analysis of simulation outcomes: mean

- Expected value $\mathbb{E}[W(t)]$ in black
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)
Analysis of simulation outcomes: distribution

► There is more information in the simulation’s output
► Estimate the distribution function of \( W(t) \) ⇒ Histogram

► Consider a grid of points \( w^{(0)}, \ldots, w^{(M)} \)
► Indicator function of the event \( w^{(m)} \leq W_j(t) < w^{(m+1)} \)

\[
I \left\{ w^{(m)} \leq W_j(t) < w^{(m+1)} \right\} = \begin{cases} 
1, & \text{if } w^{(m)} \leq W_j(t) < w^{(m+1)} \\
0, & \text{otherwise}
\end{cases}
\]

► Histogram is then defined as

\[
H \left[ t; w^{(m)}, w^{(m+1)} \right] = \frac{1}{J} \sum_{j=1}^{J} I \left\{ w^{(m)} \leq W_j(t) < w^{(m+1)} \right\}
\]

► Fraction of experiments with wealth \( W_j(t) \) between \( w^{(m)} \) and \( w^{(m+1)} \)
Distribution broadens and shifts to the right ($t = 10, 50, 100, 200$)
What is this class about?

- Analysis and simulation of **stochastic systems**
  - A system that **evolves in time** with some **randomness**

- They are usually quite **complex**  ⇒ Simulations

- We will learn how to **model** stochastic systems, e.g.,
  - \( X(t) \) Bernoulli with parameter \( p \)
  - \( W(t+1) = W(t) + 1 \), when \( X(t) = 1 \)
  - \( W(t+1) = W(t) - 1 \), when \( X(t) = 0 \)

- ... how to **analyze** their properties, e.g., \( \mathbb{E}[W(t)] = w_0 + (2p - 1)t \)

- ... and how to **interpret** simulations and experiments, e.g.,
  - Average tendency through sample average
  - Estimate probability distributions via histograms