Introduction to Random Processes

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Introductions

Class description and contents

Gambling
Who we are, where to find me, lecture times

▸ Gonzalo Mateos
▸ Associate Professor, Dept. of Electrical and Computer Engineering
▸ CSB 726, gmateosb@ece.rochester.edu
▸ http://www.ece.rochester.edu/~gmateosb

▸ Where? We meet in Gavett Hall 202
▸ When? Mondays and Wednesdays 4:50 pm to 6:05 pm

▸ My office hours, Tuesdays at 10:30 am
  ▸ Anytime, as long as you have something interesting to tell me

▸ Class website
  http://www.ece.rochester.edu/~gmateosb/ECE440.html
Teaching assistants

- Three great TAs to help you with your homework

- **Narges Mohammadi**
  - CSB 633, nmohamm4@ur.rochester.edu
  - Her office hours, **Thursdays at 11 am**

- **Chang Ye**
  - CSB 701, cye7@ur.rochester.edu
  - His office hours, **Mondays at 2 pm**
Teaching assistants

- Three great TAs to help you with your homework
- Saman Saboksayr
- CSB 701, ssaboksa@ur.rochester.edu
- His office hours, Fridays at 10:30 am
(I) Probability theory
- Random (Stochastic) processes are collections of random variables
- Basic knowledge expected. Will review in the first six lectures

(II) Calculus and linear algebra
- Integrals, limits, infinite series, differential equations
- Vector/matrix notation, systems of linear equations, eigenvalues

(III) Programming in Matlab
- Needed for homework
  [https://tech.rochester.edu/software/matlab/](https://tech.rochester.edu/software/matlab/)
- If you know programming you can learn Matlab in one afternoon
  ⇒ But it has to be one of this week’s afternoons
Homework, exams and grading

(I) **Homework sets** (10 in 15 weeks) worth 28 points
- Important and demanding part of this class
- Collaboration accepted, welcomed, and encouraged

(II) **Midterm examination** on Monday **November 1st** worth 36 points

(III) **Final take-home examination** on **December 12-14** worth 36 points
- Work independently. *This time no collaboration, no discussion*
- ECE 271 students get **10 free points**
- At least 60 points are required for passing (C grade)
- **B requires at least 75 points. A at least 92. No curve**
  ⇒ **Goal is for everyone to earn an A**
Textbooks

- Good general reference for the class
  
  
  ≫ Available online: http://www.library.rochester.edu/

- Also nice for topics including Markov chains, queuing models
  

- Both on reserve for the class in Carlson Library
Be nice

- I work hard for this course, expect you to do the same
- If you come to class, be on time, pay attention, ask
- Wear your masks, no food in class, wash your hands
- Do all of your homework
- Do not hand in as yours the solution of others (or mine)
- Do not collaborate in the exams

- A little bit of (conditional) probability ...
- Probability of getting an E in this class is 0.04
- Probability of getting an E given you skip 4 homework sets is 0.7
  ⇒ I’ll give you three notices, afterwards, I’ll give up on you
- Come and learn. Useful down the road
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Stochastic systems

- **Stochastic system:** Anything random that evolves in time
  - Time can be discrete \( n = 0, 1, 2, \ldots \), or continuous \( t \in [0, \infty) \)

- More formally, random processes assign a function to a random event
- Compare with “random variable assigns a value to a random event”
- Can interpret a random process as a collection of random variables
  - Generalizes concept of random vector to functions
  - Or generalizes the concept of function to random settings
A voice recognition system

- Random event $\sim$ word spoken. Random process $\sim$ the waveform
- Try the file speech_signals.m

![Waveform graphs for "Hi", "Good", "Bye", ‘S’](image-url)
Four thematic blocks

(I) Probability theory review (6 lectures)
   ▶ Probability spaces, random variables, independence, expectation
   ▶ Conditional probability: time \( n + 1 \) given time \( n \), future given past ...
   ▶ Limits in probability, almost sure limits: behavior as \( n \to \infty \) ...
   ▶ Common probability distributions (binomial, exponential, Poisson, Gaussian)
   ▶ Random processes are complicated entities
     ⇒ Restrict attention to particular classes that are somewhat tractable

(II) Markov chains (6 lectures)
(III) Continuous-time Markov chains (7 lectures)
(IV) Stationary random processes (8 lectures)
   ▶ Midterm covers up to Markov chains
Probability theory is a formalism to work with uncertainty
- Given a data-generating process, what are properties of outcomes?

Statistical inference deals with the inverse problem
- Given outcomes, what can we say on the data-generating process?
- CSC446 - Machine Learning, ECE442 - Network Science Analytics, CSC440 - Data Mining, ECE441 - Detection and Estimation Theory, ...
Markov chains

- **Countable set of states 1, 2, ..., At discrete time $n$, state is $X_n$**
- **Memoryless (Markov) property**
  - $\Rightarrow$ Probability of next state $X_{n+1}$ depends on current state $X_n$
  - $\Rightarrow$ But not on past states $X_{n-1}, X_{n-2}, ...$

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Tomorrow’s mood only affected by today’s mood
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

- **Of interest:** classification of states, ergodicity, limiting distributions
- **Applications:** Google’s PageRank, communication networks, queues, reinforcement learning, ...

![Diagram of Markov chain]

- $H \xrightarrow{0.2} S$
- $S \xrightarrow{0.7} H$
- $H \xrightarrow{0.8} H$
- $S \xrightarrow{0.3} S$
Continuous-time Markov chains

- **Countable** set of states 1, 2, ...  
  - Continuous-time index $t$, state $X(t)$
  - Transition between states can happen at any time
  - **Markov**: Future independent of the past given the present

- Probability of changing state in an infinitesimal time $dt$

- **Of interest**: Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions

- **Applications**: Chemical reactions, queues, epidemic modeling, traffic engineering, weather forecasting, ...
Stationary random processes

- **Continuous** time \( t \), **continuous state** \( X(t) \), not necessarily Markov
- Prob. distribution of \( X(t) \) constant or becomes constant as \( t \) grows
  \( \Rightarrow \) System has a **steady state in a random sense**

- **Of interest**: Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density

- **Applications**: Black Scholes model for option pricing, radar, face recognition, noise in electric circuits, filtering and equalization, ...
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Gambling
An interesting betting game

- There is a certain game in a certain casino in which ...
  - Your chances of winning are $p > 1/2$
- You place $1$ bets
  - (a) With probability $p$ you gain $1$; and
  - (b) With probability $1 - p$ you lose your $1$ bet
- The catch is that you either
  - (a) Play until you go broke (lose all your money)
  - (b) Keep playing forever
- You start with an initial wealth of $w_0$
- Q: Shall you play this game?
Let $t$ be a time index (number of bets placed)

Denote as $X(t)$ the outcome of the bet at time $t$
\[\Rightarrow X(t) = 1 \text{ if bet is won (w.p. } p)\]
\[\Rightarrow X(t) = 0 \text{ if bet is lost (w.p. } 1 - p)\]

$X(t)$ is called a Bernoulli random variable with parameter $p$

Denote as $W(t)$ the player’s wealth at time $t$. Initialize $W(0) = w_0$

At times $t > 0$ wealth $W(t)$ depends on past wins and losses
\[\Rightarrow \text{ When bet is won } W(t + 1) = W(t) + 1\]
\[\Rightarrow \text{ When bet is lost } W(t + 1) = W(t) - 1\]

More compactly can write $W(t + 1) = W(t) + (2X(t) - 1)$
\[\Rightarrow \text{ Only holds so long as } W(t) > 0\]
$t = 0; \ w(t) = w_0; \ \text{max}_t = 10^3; \ // \ Initialize \ variables$

% repeat while not broke up to time $\text{max}_t$

\textbf{while} (w(t) > 0) & (t < \text{max}_t) \ \textbf{do}

\hspace{1em} x(t) = \text{random}(\text{'bino'}, 1, p); \ % \ Draw \ Bernoulli \ random \ variable

\hspace{2em} \textbf{if} \ x(t) == 1 \ \textbf{then}

\hspace{3em} w(t + 1) = w(t) + b; \ % \ If \ x = 1 \ wealth \ increases \ by \ b

\hspace{2em} \textbf{else}

\hspace{3em} w(t + 1) = w(t) - b; \ % \ If \ x = 0 \ wealth \ decreases \ by \ b

\hspace{1em} \textbf{end}

\hspace{1em} t = t + 1;

\textbf{end}

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$

- Q: Shall we play?
One lucky player

She didn’t go broke. After $t = 1000$ bets, her wealth is $W(t) = 109$

⇒ Less likely to go broke now because wealth increased
Two lucky players

- After \( t = 1000 \) bets, wealths are \( W_1(t) = 109 \) and \( W_2(t) = 139 \)

\[ \Rightarrow \text{Increasing wealth seems to be a pattern} \]
Ten lucky players

- Wealths $W_j(t)$ after $t = 1000$ bets between 78 and 139

$\Rightarrow$ Increasing wealth is definitely a pattern
One unlucky player

- But this does not mean that all players will turn out as winners
  \[ \Rightarrow \text{The twelfth player } j = 12 \text{ goes broke} \]
But this does not mean that all players will turn out as winners. The twelfth player $j = 12$ goes broke.
One hundred players

- All players (except for $j = 12$) end up with substantially more money
It is not difficult to find a line estimating the average of $W(t)$

\[ \tilde{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t \quad \text{(recall } p = 0.55) \]
Where does the average tendency come from?

- Assuming we do not go broke, we can write
  \[ W(t + 1) = W(t) + \left( 2X(t) - 1 \right), \quad t = 0, 1, 2, \ldots \]

  - The assumption is incorrect as we saw, but suffices for simplicity

- Taking expectations on both sides and using linearity of expectation
  \[ \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + \left( 2\mathbb{E}[X(t)] - 1 \right) \]

- The expected value of Bernoulli \( X(t) \) is
  \[ \mathbb{E}[X(t)] = 1 \times P(X(t) = 1) + 0 \times P(X(t) = 0) = p \]

- Which yields
  \[ \Rightarrow \quad \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + (2p - 1) \]

- Applying recursively
  \[ \Rightarrow \quad \mathbb{E}[W(t + 1)] = w_0 + (2p - 1)(t + 1) \]
Gambling as LTI system with stochastic input

- Recall the evolution of wealth $W(t + 1) = W(t) + \left(2X(t) - 1\right)$

- View $W(t + 1)$ as output of LTI system with random input $2X(t) - 1$

- Recognize accumulator $\Rightarrow W(t + 1) = w_0 + \sum_{\tau=0}^{t} \left(2X(\tau) - 1\right)$
  - Useful, a lot we can say about sums of random variables

- Filtering random processes in signal processing, communications, . . .
Numerical analysis of simulation outcomes

- For a more accurate approximation analyze simulation outcomes
- Consider $J$ experiments. Each yields a wealth history $W_j(t)$
- Can estimate the average outcome via the sample average $\bar{W}_J(t)$

$$\bar{W}_J(t) := \frac{1}{J} \sum_{j=1}^{J} W_j(t)$$

- Do not confuse $\bar{W}_J(t)$ with $\mathbb{E}[W(t)]$
  - $\bar{W}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[W(t)]$ is a property of the random variable $W(t)$
  - We will see later that for large $J$, $\bar{W}_J(t) \to \mathbb{E}[W(t)]$
Analysis of simulation outcomes: mean

- Expected value $\mathbb{E}[W(t)]$ in black
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)
Analysis of simulation outcomes: distribution

- There is more information in the simulation’s output
- Estimate the distribution function of $W(t)$ \(\Rightarrow\) Histogram
- Consider a grid of points $w(0), \ldots, w(M)$
- Indicator function of the event $w(m) \leq W_j(t) < w(m+1)$
  \[
  \mathbb{I}\left\{w(m) \leq W_j(t) < w(m+1)\right\} = \begin{cases} 
  1, & \text{if } w(m) \leq W_j(t) < w(m+1) \\
  0, & \text{otherwise}
  \end{cases}
  \]
- Histogram is then defined as
  \[
  H\left[t; w^{(m)}, w^{(m+1)}\right] = \frac{1}{J} \sum_{j=1}^{J} \mathbb{I}\left\{w(m) \leq W_j(t) < w(m+1)\right\}
  \]
- Fraction of experiments with wealth $W_j(t)$ between $w^{(m)}$ and $w^{(m+1)}$
Histogram

- Distribution broadens and shifts to the right ($t = 10, 50, 100, 200$)
What is this class about?

- Analysis and simulation of **stochastic systems**
  - A system that *evolves in time* with some *randomness*

- They are usually quite **complex**  ⇒ Simulations

- We will learn how to **model** stochastic systems, e.g.,
  - $X(t)$ Bernoulli with parameter $p$
  - $W(t + 1) = W(t) + 1$, when $X(t) = 1$
  - $W(t + 1) = W(t) - 1$, when $X(t) = 0$

- ... how to **analyze** their properties, e.g., $\mathbb{E}[W(t)] = w_0 + (2p - 1)t$

- ... and how to **interpret** simulations and experiments, e.g.,
  - Average tendency through sample average
  - Estimate probability distributions via histograms