Chapter 11

Sections 11-1 and 11-2: Fourier Series

For each of the waveforms in Problems 11.1 through 11.10:

(a) Determine if the waveform has dc, even, or odd symmetry.

(b) Obtain its cosine/sine Fourier series representation.

(c) Convert the representation to amplitude/format and plot the line spectra for the first five non-zero terms.

(d) Use MATLAB® software to plot the waveform using a truncated Fourier series representation with $n_{max} = 100$.

Problem 11.1 Waveform in Fig. P11.1 with $A = 10$.

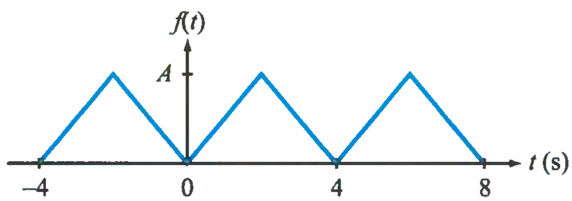


Figure P11.1: Waveform for Problem 11.1.

Solution:

(a) The waveform has even symmetry.

(b) Period $T = 4$ s

$$\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2} \text{ rad/s}$$

$$f(t) = \begin{cases} -5t & -2\text{ s} \le t \le 0 \\ 5t & 0\text{ s} \le t \le 2\text{ s} \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$= \frac{1}{4}\left[\int_{-2}^{0} -5t\,dt + \int_{0}^{2} 5t\,dt\right]$$

$$= \frac{1}{4}\left[\frac{5}{2}\times 4 + \frac{5}{2}\times 4\right] = 5$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{1}{2}\left[\int_{-2}^{0} -5t\cos\left(\frac{n\pi}{2}t\right)\,dt + \int_{0}^{2} 5t\cos\left(\frac{n\pi}{2}t\right)\,dt\right]$$

$$= \frac{20}{n^2\pi^2}[\cos(n\pi) - 1]$$

$$b_n = 0 \qquad \text{(even symmetry)}$$

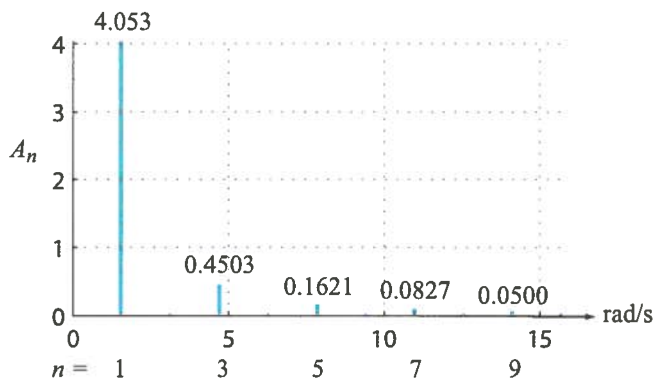
$$f(t) = 5 + \sum_{n=1}^{\infty} \frac{20}{n^2\pi^2}[\cos(n\pi) - 1]\cos\left(\frac{n\pi}{2}t\right)$$

(c) amplitude format

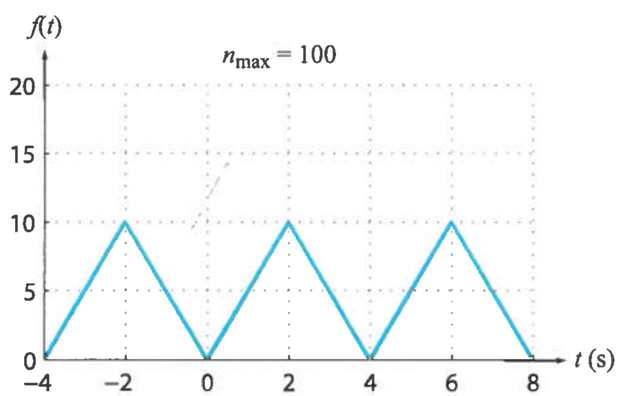
$$a_0 = 5$$

$$A_n = |a_n| = \begin{cases} \dfrac{40}{n^2\pi^2} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}$$

$$\phi_n = \begin{cases} 180^\circ & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}$$



(d)



Problem 11.2 Waveform in Fig. P11.2 with $A = 4$.

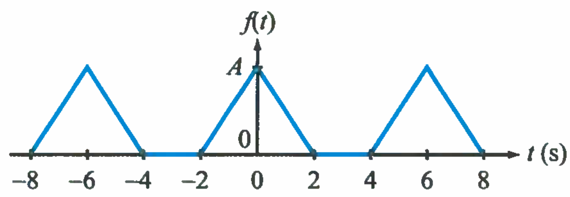


Figure P11.2: Waveform for Problem 11.2.

Solution:

(a) Waveform has even symmetry.

(b) Period $T = 6 \implies \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$

$$f(t) = \begin{cases} 0 & -3 \leq t < -2 \\ 2(t+2) & -2 \leq t \leq 0 \\ -2(t-2) & 0 < t \leq 2 \\ 0 & 2 \leq t \leq 3 \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$
$$= \frac{1}{6}\int_{-3}^{3} f(t)\,dt$$
$$= \frac{1}{6}\left[\int_{-2}^{0} 2(t+2)\,dt + \int_0^2 -2(t-2)\,dt\right]$$
$$= \frac{4}{3}$$

$$a_n = \frac{2}{T}\int_{-2}^{2} f(t)\cos\left(n\frac{\pi}{3}t\right)\,dt$$
$$= \frac{1}{3}\left[\int_{-2}^{0} 2(t+2)\cos\left(\frac{n\pi}{3}t\right)\,dt - 2\int_0^2 2(t-2)\cos\left(\frac{n\pi}{3}t\right)\,dt\right]$$
$$= \frac{12}{n^2\pi^2}\left[1-\cos\left(\frac{2\pi}{3}n\right)\right]$$

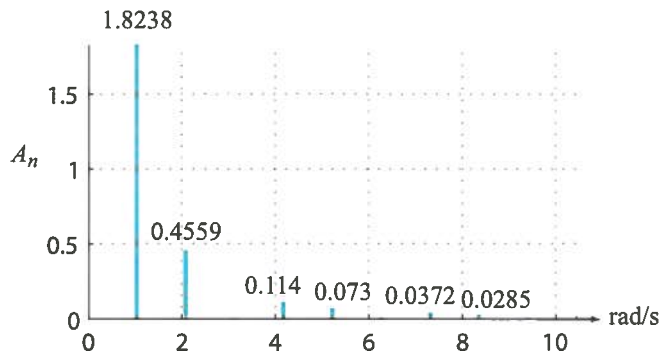
$$b_n = 0 \qquad \text{(even symmetry)}$$

$$f(t) = \frac{4}{3} + \sum_{n=1}^{\infty}\frac{12}{n^2\pi^2}\left[1-\cos\left(\frac{2\pi}{3}n\right)\right]\cos\left(\frac{n\pi}{3}t\right)$$

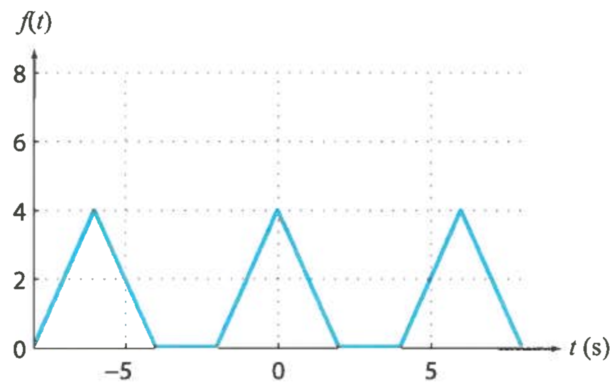
(c) amplitude format

$$A_n = \sqrt[+]{a_n^2 + b_n^2} = \frac{12}{n^2\pi^2}\left|1-\cos\left(\frac{2\pi}{3}n\right)\right|$$

$$\phi_n = 0 \qquad \text{for all } n$$


1.8238
1.5
A_n
1
0.4559
0.5
0.114
0.073
0.0372
0.0285
0
0
2
4
6
8
10
rad/s

(d)


f(t)
8
6
4
2
0
−5
0
5
t (s)

Problem 11.10 Waveform in Fig. P11.10 with $A = 20$.

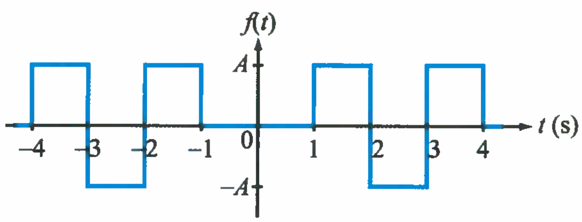


Figure P11.10: Waveform for Problem 11.10.

Solution:

(a) Even symmetry

(b) $T = 5$ s

$$\omega_0 = \frac{2\pi}{5} \text{ rad/s}$$

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 20 & 1 \le t \le 2 \\ -20 & 2 \le t \le 3 \\ 20 & 3 \le t \le 4 \\ 0 & 4 \le t \le 5 \end{cases}$$

$$a_0 = \frac{1}{5}\left[\int_1^2 20\,dt - \int_2^3 20\,dt + \int_3^4 20\,dt\right]$$
$$= 4\left[t\big|_1^2 - t\big|_2^3 + t\big|_3^4\right]$$
$$= 4$$

$$a_n = \frac{2\times 20}{5}\left[\int_1^2 \cos\left(\frac{2\pi nt}{5}\right)dt - \int_2^3 \cos\left(\frac{2\pi nt}{5}\right)dt + \int_3^4 \cos\left(\frac{2\pi nt}{5}\right)dt\right]$$
$$= \frac{20}{n\pi}\left[2\sin\left(\frac{4\pi n}{5}\right) - 2\sin\left(\frac{6\pi n}{5}\right) - \sin\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{8\pi n}{5}\right)\right]$$

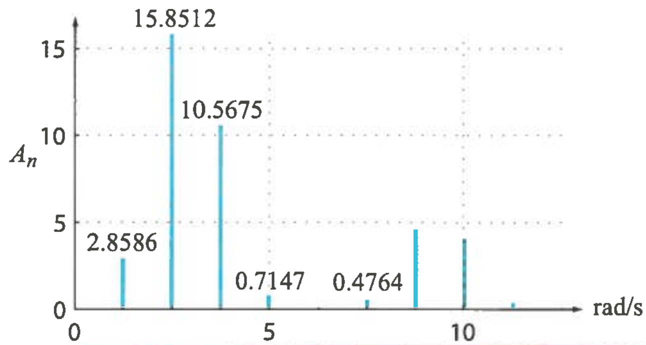
$$b_n = 0 \qquad \text{(even symmetry)}$$

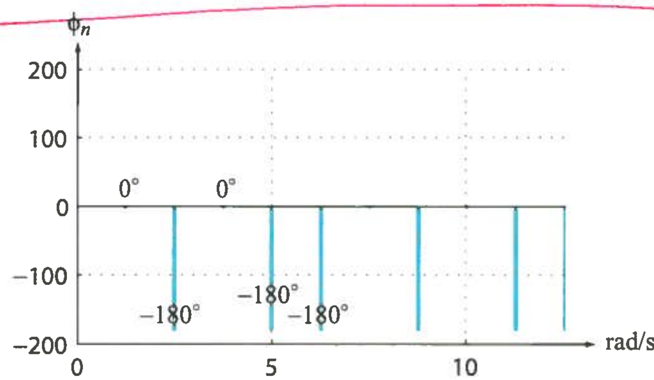
$$f(t) = 4 + \sum_{n=1}^{\infty} \frac{20}{n\pi}\left[2\sin\left(\frac{4\pi n}{5}\right) - 2\sin\left(\frac{6\pi n}{5}\right) - \sin\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{8\pi n}{5}\right)\right] \cdot \cos\left(\frac{2\pi nt}{5}\right)$$

(c) Amplitude format

$$A_n = +\sqrt{a_n^2 + b_n^2} = \frac{20}{n\pi}\left|2\sin\left(\frac{4\pi n}{5}\right) - 2\sin\left(\frac{6\pi n}{5}\right) - \sin\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{8\pi n}{5}\right)\right|$$

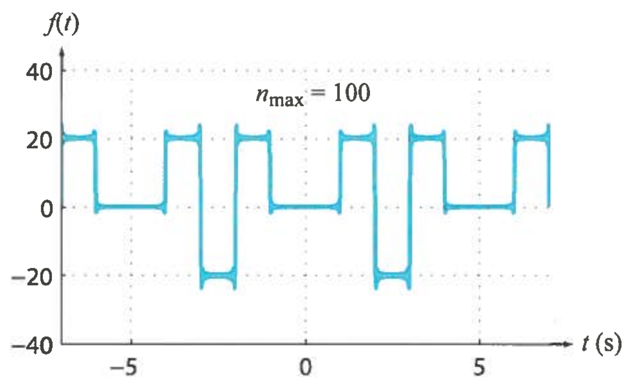
$$\phi_n = \begin{cases} 0^\circ & \text{for } a_n > 0 \\ 180^\circ & \text{for } a_n < 0 \end{cases}$$


15.8512
15
10.5675
10
A_n
5
2.8586
0.7147
0.4764
0
0
5
10
rad/s


ϕ_n
200
100
0°
0°
0
−100
−180°
−180°
−180°
−200
0
5
10
rad/s

(d)

$\phi_n =$


f(t)
40
$n_{max} = 100$
20
0
−20
−40
−5
0
5
t (s)

Problem 11.16 The Fourier series of the periodic waveform shown in Fig. P11.16(a) is given by

$$f_1(t) = 10 - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi t}{2}\right).$$

Determine the Fourier series of the waveform in Fig. P11.16(b).

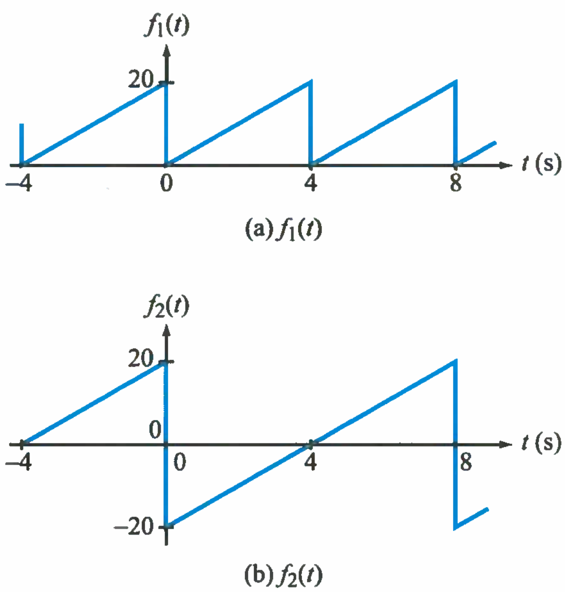


Figure P11.16: Waveforms of Problem 11.16.

Solution: $f_2(t)$ appears to be a shifted-down version of $f_1(t)$ with the following differences:

In the case of $f_1(t)$: its period is $T_1 = 4$ s, hence $\omega_{0_1} = \frac{\pi}{2}$ rad/s. Its peak-to-peak value $= 20$.

In the case of $f_2(t)$: its period is $T_2 = 8$ s, hence $\omega_{0_2} = \frac{\pi}{4}$ rad/s. Its peak-to-peak value $= 40$.

$$\therefore \quad f_2(t) = 2[f_1(t) - 10] \quad \text{with } f_1(t) \text{ period modified to be that of } f_2(t)$$

$$\therefore \quad f_2(t) = -\frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi t}{4}\right).$$

10

Section 11-3: Circuit Applications

Problem 11.17 The voltage source $v_s(t)$ in the circuit of Fig. P11.17 generates a square wave (waveform #1 in Table 11-2) with $A = 10$ V and $T = 1$ ms.

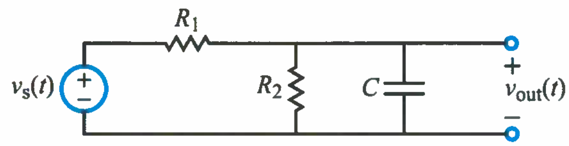


Figure P11.17: Circuit for Problem 11.17.

(a) Derive the Fourier series representation of $v_{out}(t)$.
(b) Calculate the first five terms of $v_{out}(t)$ using $R_1 = R_2 = 2$ kΩ, $C = 1$ μF.
(c) Plot $v_{out}(t)$ using $n_{max} = 100$.

Solution: According to Table 11-2,

$$v_s(t) = \sum_{n=1}^{\infty} \frac{40}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(2\pi \times 10^3 nt)$$
$$\omega_0 = 2\pi \times 10^3$$

Then,

$$a_0 = 0$$
$$b_n = 0$$
$$a_n = \frac{40}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

In phasor domain

$$\mathbf{V}_s = \sum_{n=1}^{\infty} A_n e^{j\phi_n}$$
$$A_n \angle\phi_n = a_n - jb_n = \frac{40}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

Next, we compute the transfer function of the circuit:

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_s} = \frac{\mathbf{Z}_{out}}{\mathbf{Z}_{out} + R_1}$$
$$\mathbf{Z}_{out} = \frac{R_2 \frac{1}{j\omega C}}{R_2 + \frac{1}{j\omega C}} = \frac{R_2}{1 + j\omega C R_2}$$

Thus,

$$\mathbf{H}(\omega) = \frac{R_2}{(R_1 + R_2) + j\omega C R_1 R_2}.$$

Substituting the values of circuit elements into $H(\omega)$,

$$\mathbf{H}(\omega) = \frac{2\text{k}}{4\text{k} + j4\omega} = \frac{2\text{k}}{\sqrt{(4\text{k})^2 + (4\omega)^2}} e^{-j\tan^{-1}(\omega \times 10^{-3})}$$

$$\therefore \quad \mathbf{V}_{\text{out}} = \sum_{n=1}^{\infty} A_n \, \mathbf{H}(\omega = n\omega_0)$$

$$= \sum_{n=1}^{\infty} A_n \frac{2\text{k}}{\sqrt{(4\text{k})^2 + (4\omega_0 n)^2}} e^{-j\tan^{-1}(n\omega_0 \times 10^{-3})}$$

In time domain:

$$v_{\text{out}}(t) = \sum_{n=1}^{\infty} \mathfrak{Re}\left\{ A_n \frac{2\text{k}}{\sqrt{(4\text{k})^2 + (4\omega_0 n)^2}} e^{-j\tan^{-1}(n\omega_0 \times 10^{-3})} e^{j\omega_0 nt} \right\}$$

$$= \sum_{n=1}^{\infty} \frac{40}{n\pi} \sin\left(\frac{n\pi}{2}\right) \frac{2000}{\sqrt{(4\text{k})^2 + (4\omega_0 n)^2}} \cos[\omega_0 nt - \tan^{-1}(2\pi n)]$$ 5

(b) The first 5 terms of $v_{\text{out}}(t)$ are

$n = 1 \longrightarrow 1.0006\cos(\omega_0 t - 1.413)$ 1

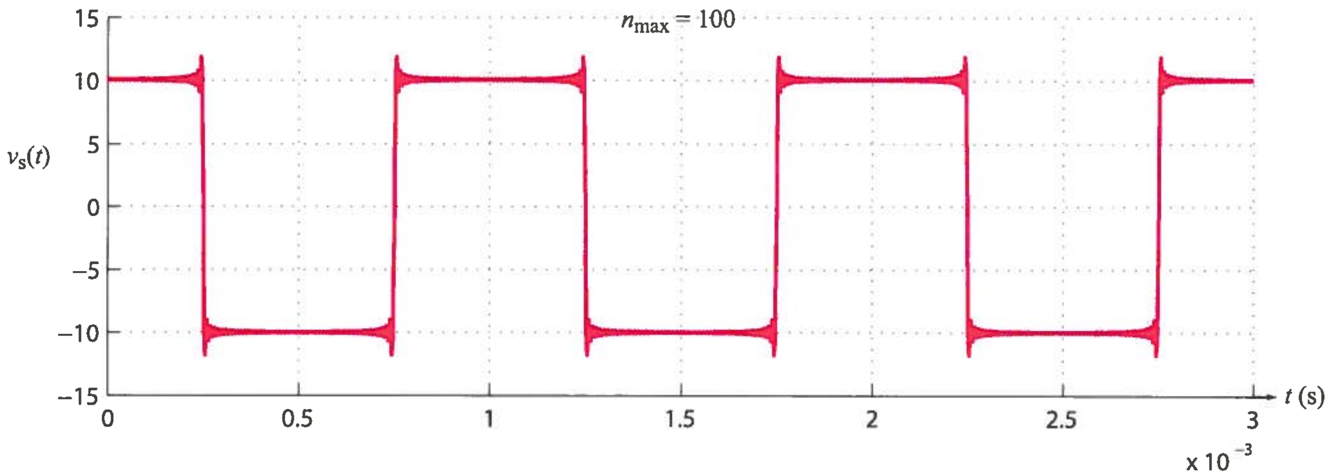
$n = 2 \longrightarrow 0$ 1

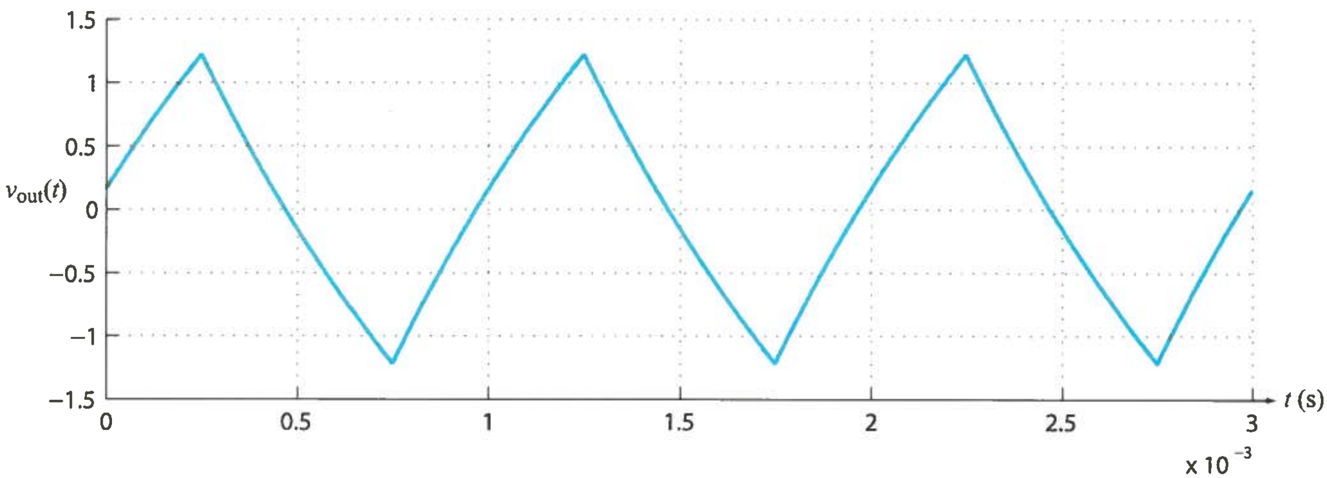
$n = 3 \longrightarrow -0.1124\cos(3\omega_0 t - 1.5178)$ 1

$n = 4 \longrightarrow 0$ 1

$n = 5 \longrightarrow 0.0405\cos(5\omega_0 t - 1.539)$ 1

(c)


$n_{max} = 100$
15
10
5
0
−5
−10
−15
$v_S(t)$
0
0.5
1
1.5
2
2.5
3
t (s)
x 10^-3


1.5
1
0.5
0
−0.5
−1
−1.5
$v_{out}(t)$
0
0.5
1
1.5
2
2.5
3
t (s)
x 10^-3

Section 11-4: Average Power

Problem 11.28 The voltage across the terminals of a certain circuit and the current entering into its (+) voltage terminal are given by

$$v(t) = [4 + 12\cos(377t + 60^\circ) - 6\cos(754t - 30^\circ)] \text{ V},$$
$$i(t) = [5 + 10\cos(377t + 45^\circ) + 2\cos(754t + 15^\circ)] \text{ mA}.$$

Determine the average power consumed by the circuit, and the ac power fraction.

Solution: Application of Eq. (11.43) yields

$$P_{av} = \left[4 \times 5 + \frac{12 \times 10}{2}\cos(60^\circ - 45^\circ) - \frac{6 \times 2}{2}\cos(-30^\circ - 15^\circ)\right] \times 10^{-3}$$
$$= (20 + 57.96 - 4.24) \times 10^{-3} = 73.72 \text{ mW}$$

(6)

$$\text{ac fraction} = \frac{57.96 + 4.24}{73.72} = 84.37\%.$$

(4)

Problem 11.29 The current flowing through a 2-kΩ resistor is given by

$$i(t) = [5 + 2\cos(400t + 30^\circ) + 0.5\cos(800t - 45^\circ)] \text{ mA}.$$

Determine the average power consumed by the resistor, as well as the ac power fraction.

Solution: Voltage across the resistor is $v(t) = 2000 \times i(t)$.

$$v(t) = 10 + 4\cos(400t + 30^\circ) + \cos(800t - 45^\circ) \text{ V}.$$

Application of Eq. (11.43) yields

$$P_{av} = \left[5 \times 10 + \frac{2 \times 4}{2}\cos(30^\circ - 30^\circ) + \frac{0.5}{2}\cos(-45^\circ + 45^\circ)\right] \times 10^{-3}$$

$$= (50 + 4 + 0.5) = 54.5 \text{ mW}$$

0.25 = 54.25 mW (6)

$$\text{ac power fraction} = \frac{4.5}{54.5} = 8.26\%.$$

4.25 / 54.25 = 7.83% (4)

Sections 11-5 and 11-6: Fourier Transform

For each of the waveforms in Problems 11.33 through 11.42, determine the Fourier transform.

Problem 11.33 Waveform in Fig. P11.33 with $A = 5$ and $T = 3$ s.

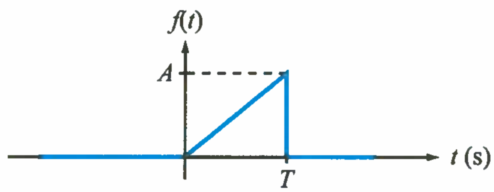


Figure P11.33: Waveform for Problem 11.33.

Solution: The waveform can be expressed as

$$f(t) = \begin{cases} 0 & t \leq 0 \\ \frac{A}{T}\,t & 0 \leq t \leq T \\ 0 & t \geq T. \end{cases}$$

Applying the Fourier transform definition to $f(t)$,

$$\begin{aligned}
\mathbf{F}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\
&= \int_0^T \frac{A}{T}\, te^{-j\omega t}\, dt \\
&= \frac{A}{T}\left[\frac{jte^{-j\omega t}}{\omega}\bigg|_0^T - \frac{j}{\omega}\int_0^T e^{-j\omega t}\, dt\right] \\
&= \frac{A}{T\omega^2}\left[e^{-j\omega T}(1 + j\omega T) - 1\right].
\end{aligned}$$

Substituting for the values of A and T:

$$\mathbf{F}(\omega) = \frac{5}{3\omega^2}[(1 + j3\omega)e^{-j3\omega} - 1].$$

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Problem 11.34 Waveform in Fig. P11.34 with $A = 10$ and $T = 6$ s.

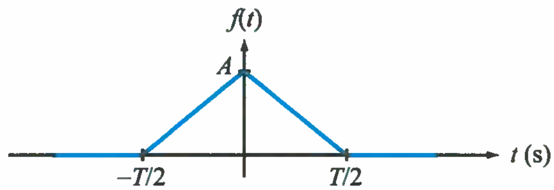

Figure P11.34: Waveform for Problem 11.34.

Solution: The waveform can be expressed as

$$f(t) = \begin{cases} 0 & t < -\dfrac{T}{2} \\ \dfrac{2A}{T}\left(t + \dfrac{T}{2}\right) & -\dfrac{T}{2} \le t \le 0 \\ -\dfrac{2A}{T}\left(t - \dfrac{T}{2}\right) & 0 \le t \le \dfrac{T}{2} \\ 0 & t > \dfrac{T}{2}. \end{cases}$$

Applying the definition of Fourier transform to $f(t)$,

$$\begin{aligned} \mathbf{F}(\omega) &= \int_{-T/2}^{0} \frac{2A}{T}\left(t + \frac{T}{2}\right) e^{-j\omega t}\,dt + \int_{0}^{T/2} -\frac{2A}{T}\left(t - \frac{T}{2}\right) e^{-j\omega t}\,dt \\ &= \left[\frac{jA}{\omega}\,e^{-j\omega t}\right]_{-T/2}^{0} + \left[\frac{jA}{\omega}\,e^{-j\omega t}\right]_{0}^{T/2} + \left[\frac{2jAt}{\omega T}\,e^{-j\omega t}\right]_{-T/2}^{0} - \left[\frac{2jAt}{\omega T}\,e^{-j\omega t}\right]_{0}^{T/2} \\ &\quad - \int_{-T/2}^{0} \frac{2Aj}{\omega T}\,e^{-j\omega t}\,dt + \int_{0}^{T/2} \frac{2Aj}{\omega T}\,e^{-j\omega t}\,dt \end{aligned}$$

$$\mathbf{F}(\omega) = \frac{4A}{T\omega^2}\left[1 - \cos\left(\frac{\omega T}{2}\right)\right].$$

Substituting for A and T,

$$\mathbf{F}(\omega) = \frac{20}{3\omega^2}\,[1 - \cos(3\omega)].$$

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Problem 11.35 Waveform in Fig. P11.35 with $A = 12$ and $T = 3$ s.

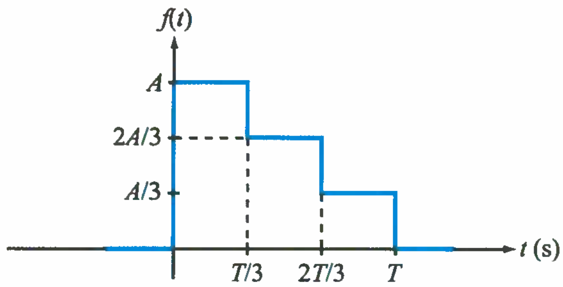

Figure P11.35: Waveform for Problem 11.35.

Solution: The waveform can be expressed as

$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 \le t \le T/3 \\ 2A/3 & T/3 \le t \le 2T/3 \\ A/3 & 2T/3 \le t \le T \\ 0 & t > T. \end{cases}$$

Applying the Fourier transform definition to $f(t)$,

$$\mathbf{F}(\omega) = \int_0^{T/3} Ae^{-j\omega t}\,dt + \int_{T/3}^{2T/3} \frac{2A}{3}\,e^{-j\omega t}\,dt + \int_{2T/3}^{T} \frac{A}{3}\,e^{-j\omega t}\,dt$$

$$= \left[\frac{jA}{\omega}\,e^{-j\omega t}\right]_0^{T/3} + \left[\frac{2jA}{3\omega}\,e^{-j\omega t}\right]_{T/3}^{2T/3} + \left[\frac{jA}{3\omega}\,e^{-j\omega t}\right]_{2T/3}^{T}$$

$$\mathbf{F}(\omega) = \frac{2A}{3\omega}\sin\left(\frac{\omega T}{6}\right)\,[3e^{-j\omega T/6} + 2e^{-j\omega T/2} + e^{-j\omega 5T/6}].$$

Substituting for A and T,

$$\Rightarrow \quad \mathbf{F}(\omega) = \frac{8}{\omega}\sin\left(\frac{\omega}{2}\right)\,[3e^{-j\omega/2} + 2e^{-j3\omega/2} + e^{-j5\omega/2}].$$

10